

Bewley-Hugget-Aiyagari Models

Basic Properties and Numerical solution methods

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This script is meant to provide an introduction into quantitative modelling by discussing the properties and solution methods for models that follow the Bewley-Hugget-Aiyagari type economy. The main focus is on students who start to learn about quantitative modelling in economics, but especially some of the mathematical and numerical properties might also be interesting for more advanced users.

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1 Introduction

In economic theory, representative-agent models have traditionally been the cornerstone of equilibrium analysis. These models assume that individuals' heterogeneity does not matter, since the aggregation of all agents yields an economy that can be thought of as being inhabited by a single type of agent. While this abstraction offers several advantages, including tractability and analytical solutions, macroeconomic literature has become increasingly concerned with its shortcomings, particularly with its inability to replicate crucial features of household distributions. This limitation has become especially apparent given the growing availability of high-quality micro-data, which has allowed economists to improve their theories and select sensible parameters for their models to match important statistics observed in the real economy.

Consequently, in the past three decades, there has been a growing emphasis in economic literature on developing and utilizing models which incorporate heterogeneous agents. These models consider the influence of individual-specific factors, such as income, which are typically determined by external shocks. By moving away from analyzing the decisions of a single representative agent economists can capture the distributional effects arising from a diverse set of consumers whose behavior varies based on their individual states. This approach enables economists to replicate various distributional characteristics observed in household data, including wealth, income, and consumption patterns. By accounting for the rich heterogeneity present in the data, the equilibrium analysis of these models can differ drastically from the representative-agent approach. This not only provides new insights into economic channels, but also improves the quantitative evaluation of government policies, especially when looking at distributional outcomes such as various dimensions of inequality.

Nowadays, there exist several different approaches for matching empirical moments with quantitative models. The first choice involves the modelling of time which can be formulated as being discrete or continuous. While the former is more common in modern macroeconomics, there still exist notable studies which employ a continuous time formulation. Luttmer (2007), for example, studies balanced growth that is consistent with the observed size distribution of firms. Benhabib et al. (2011) analyze the dynamics of the distribution of wealth, and Moll (2014) studies the effect of financial frictions on capital misallocation. Also, the introduction of individual heterogeneity can be combined with aggregate shocks that affect everyone. For example, Krusell and Smith (1998) consider both idiosyncratic heterogeneity and movements in aggregate productivity to develop a generalized stochastic growth model.

The focus of this thesis, however, is on models which employ a discrete time formulation

without aggregate shocks. Among the seminal contributions to this strand of economic literature was the introduction of a large number of agents who receive idiosyncratic income shocks that are uninsured as in the models by Bewley (1987). Subsequent works by Hugget (1993) and Aiyagari (1994) further expanded on the idea, and the class of models which builds upon their contributions is now commonly known as a Bewley-Hugget-Aiyagari (BHA) type economy. Models of this type have since been used to analyse a variety of topics. Castañeda et al. (2003), for example, were able to almost exactly account for the U.S. earnings and wealth inequality. Pijoan-Mas (2006) analysed the effects of longer working hours and Conesa et al. (2009) provided a welfare analysis for an optimal tax on capital and labor income. In other dimensions, Chatterjee et al. (2007) and Livshits et al. (2007) introduced unsecured borrowing and equilibrium default to improve studies of the credit market, while Quadrini (2000) investigated entrepreneurship.

Each of these studies uses equilibrium analysis to explore model dynamics and compare outcomes. While some studies focus solely on steady-state mechanisms, others employ counterfactual analyses to examine a range of scenarios and better understand the behavior of the model. Given the importance of these models in generating insights about macroeconomic phenomena, it is crucial to understand their mathematical properties. For example, the uniqueness and even existence of a general equilibrium are not always guaranteed. Furthermore, the computation of an equilibrium, if it exists, often relies on numerical solution methods and approximations of individual decisions. Since many key conclusions drawn from quantitative macroeconomic models rely on these properties, a sound mathematical characterization of BHA models is essential for interpreting the results.

Consequently, the study of large dynamic economies has a long tradition in the literature. Jovanovic and Rosenthal (1988), for example, present a sequential-game model with a continuum of agents and establish some criteria for the existence of general and stationary equilibria. Similarly, Stokey et al. (1989b) provide an overview of existing results on stochastic dynamic programming problems. Furthermore, the characterization of equilibrium properties has extended to studies of comparative statics which discuss how equilibria respond to a change in fundamentals like shocks to preferences or production parameters (e.g., Corchón, 1994; Acemoglu and Jensen, 2013). In particular, Acemoglu and Jensen (2015) consider infinite-horizon economies populated by a continuum of agents subject to idiosyncratic shocks. Hence, their results can be applied to a large set of frameworks in the spirit of BHA models.

Despite significant progress in analyzing large dynamic economies, qualitative discussions of model results in economics still frequently rely on strong assumptions, such as

restricting shock distributions or functional forms. Therefore, economists typically use numerical algorithms and approximations based on dynamic optimization to obtain more general results. Early foundations for these algorithms can be found in the seminal work by Bellman (1957). Also, the formulation of infinite-horizon models where discounting can give rise to stationary solutions has proved especially influential (see, e.g., Blackwell, 1965). However, the properties of numerical solutions are oftentimes not as well understood as those of the approximated equilibria. This can not only lead to misleading results, but also prove very problematic. Hatchondo et al. (2010), for example, provide a practical example of how numerical errors can lead to spurious interest rate movements. Thus, Santos and Peralta-Alva (2005) and Peralta-Alva and Santos (2014), among others, study accuracy properties of simulations and numerical solutions of stochastic dynamic models. Similarly, Kirkby (2017) discusses theoretical results on numerical error bounds. Also, the theory developed in Kirkby (2019) provides sufficient conditions under which the solution to the numerical algorithm converges to the true solution of the BHA model itself.

The aim of this lecture is to provide an overview of the most important results regarding the theoretical and numerical characteristics of BHA models. The findings are mainly based on the results presented in Acemoglu and Jensen (2015) and in the third chapter of the PhD thesis by Kirkby (2014). The first is concerned with the existence of general equilibria in the BHA class of models, and the latter establishes important properties for the numerical algorithms which are used to compute these equilibria. The chapter from Kirkby (2014) is the basis for Kirkby (2017, 2019) and here, we will mostly refer to one of these papers instead of the thesis when citing a specific result, since they have been published in peer reviewed journals.

2 Theoretical Properties of BHA Models

2.1 Illustrative Example

There are several features that can be found in a basic BHA-Model following the seminal formulations in Bewley (1987), Hugget (1993) and Aiyagari (1994). To illustrate the basic model ingredients, Aiyagari (1994) provides a good starting point. The model economy is populated by a continuum of measure 1 of infinitely lived households who are endowed with assets and have to decide on how much to consume and how much to save in every period. The production of the consumption good takes place in a representative firm with capital and labor supply as production inputs.

The Individual's Decision Problem. Individuals derive per-period utility from consumption, $u(c)$, where $u(\cdot)$ is the so-called utility function which satisfies some natural assumptions² and c the consumption level. Future utility flows are discounted at a rate $\beta \in (0, 1)$. The individual's decision problem is to maximize their expected lifetime utility given by

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}. \quad (2.1)$$

To finance their consumption streams, individuals provide labor to earn income and can get a return on their savings. At any given period t , the amount of savings and consumption of a given consumer has to be equal to their endowment in this period. Hence, the optimization problem is subject to a budget constraint which has to hold every period:

$$c_t + a_{t+1} = wh_t + (1 + r)a_t, \quad (2.2)$$

where c_t , a_t and h_t denote period t consumption, asset holdings and labor supply, and w and r are the wage rate and the return on assets. Labor supply is assumed to follow an idiosyncratic stochastic process whose realizations have a bounded support given by $[h_{min}, h_{max}]$, with $h_{min} > 0$.³ Note that a_t could be negative at any given period, indicating that the household is borrowing to smooth consumption. However, there exists a borrowing limit, b , and we impose the natural restriction that consumption should be strictly positive in every period, i.e. $\forall t \in \mathbb{N}_0 : c_t > 0$.

Combining all these ingredients, the individual optimization problem can be specified as

$$\begin{aligned} & \max \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \\ & \text{subject to} \\ & c_t + a_{t+1} = wh_t + (1 + r)a_t, \\ & c_t \geq 0, \quad a_t \geq -b \quad \text{almost surely (a.s)}. \end{aligned} \quad (2.3)$$

Aggregation. We can see that the individual decision problem is ex-ante identical for all households. However, during their life-time, different realizations of the labor supply h will lead to diverging paths. Since we have a continuum of agents, a law of large numbers for

²In the standard cases, the function will be differentiable, monotonically increasing, and concave.

³This represents the idiosyncratic shocks mentioned in the introduction. Ex-ante all households are identical but during their life-time they might experience different shock histories which will lead to different decisions and potentially vastly differing asset holding.

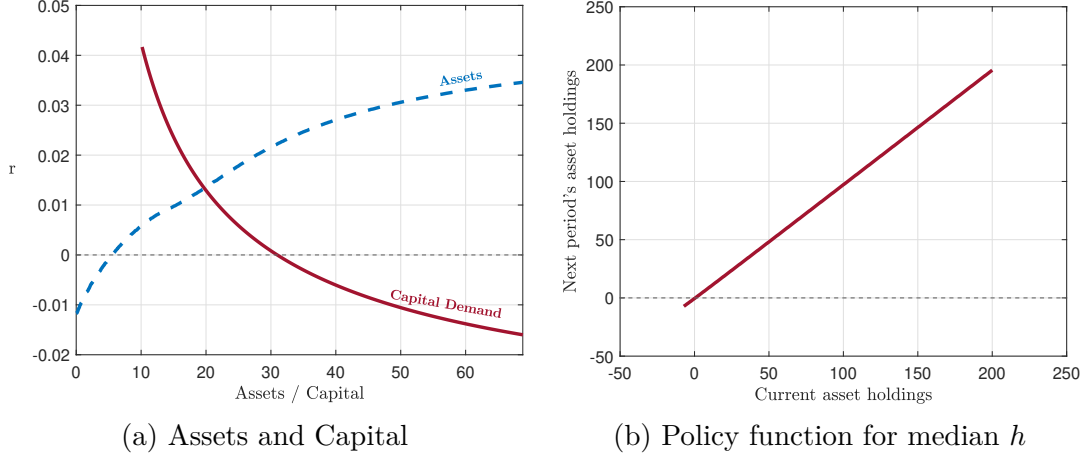


Figure 1: Illustration of model mechanisms

stochastic processes will provide us with a measure of realizations of h for every period t . Using this measure we can aggregate all the individual decisions to compute the aggregate savings in the economy, A_t . For example, let the possible states be denoted by the set S and let μ_t be the measure over states for period t . Then, aggregate savings in period t are given by

$$A_t = \int_S a_t(s) \mu_t(ds)$$

Similarly, we can calculate aggregate consumption, C_t , and aggregate labor supply, L_t . These aggregates are important in characterizing the equilibrium of the model as discussed in Section 2.2.

Production. In every period, the consumption good is produced using total labor supply, L_t , and capital, K_t . Let the production function be denoted by $F(K_t, L_t)$. Then, the aggregate budget constraint of the economy is given by

$$C_t + K_{t+1} = F(K_t, L_t) - wL_t - rK_t, \quad (2.4)$$

where the rental rate of capital, r , is the same as the return on household savings. The demand for capital is determined by profit maximization and yields the first-order condition $r = \delta F(K_t, L_t)/\delta K_t$. Hence, for any given rate of return r the capital demand from the production side of the economy might differ from aggregate savings of the households. In equilibrium, however, we have $K_t = A_t$.⁴

Illustration. Figure 1 illustrates some of the mechanisms of the model solution. The parameterization used for creating this figures are described in appendix C. Part 1a displays

⁴See Section 2.2 for the definition of an equilibrium.

the total assets held by households and the capital demand of the production sector as a function of the interest rate r . The plot reveals an inverse relationship between capital demand and asset holdings, where higher interest rates lead to increased asset holdings but decreased capital demand. Naturally, higher returns on savings induce consumers to increase their assets, while higher prices for renting capital induces firms to reduce their demand. Notably, even without discussing the equilibrium of the economy, the plot clearly shows a unique level of r at which capital demand equals asset holdings.

Part 1b presents an example of a policy function for a household with median labor endowment. For any given period, the optimal decision for next period's savings will depend on the current state. This is represented by a policy function. In this case, the optimal consumption decision is linear with respect to current asset holdings (keeping fixed the labor supply and thereby income).

2.2 Characterizing BHA-Models

The example above already shows some of the basic ingredients of BHA-Models. For the subsequent analyses of the characteristics of these kinds of models it is convenient to provide a general definition. First, however, we need to introduce the notion of Markov processes which govern the exogenous shocks in BHA-Models.

Definition 2.1 (Markov Process). Let (Z, \mathcal{Z}) be a measurable space. A transition function is a mapping $P_z : Z \times \mathcal{Z} \rightarrow [0, 1]$ such that

- (i). For each $z \in Z$, $P_z(z, \cdot)$ is a probability measure on (Z, \mathcal{Z}) .
- (ii). For each $A \in \mathcal{B}(Z)$, $P_z(\cdot, A)$ is a \mathcal{Z} -measurable function.

A Markov process is a stochastic process on (Z, \mathcal{Z}) whose state next period evolves according to P_z .

Remark. The interpretation of a Markov process is that $P_z(a, A)$ provides the probability that next period's shock z_{t+1} lies in the set A given that the current shock was a , i.e.:

$$P_z(a, A) = Pr(z_{t+1} \in A | z_t = a)$$

Note that this formulation means that next period's state only depends on current realizations while being independent of the history of the shock. This property of being a 'memoryless' process is called the Markov property and is usually used together with the notion of filtrations to define a Markov process in a formal mathematical sense.

Also, for any \mathcal{Z} -measurable function f we can define the operator

$$(Tf)(z) = \int f(z') P_z(z, dz'),$$

and interpret $(Tf)(z)$ as the expected value of f next period given that current period's realization is z , i.e.: $\mathbb{E}[f(z_{t+1}) | z_t = z]$.

Since BHA models simulate dynamic forward-looking decision making, we are often interested in the distribution over the drawn shocks at any given period. In particular, we are often interested in how the shock distribution evolves from one period to another and if there might be cases in which it stays the same.

Definition 2.2 (Invariant Distribution). Let P_z be the transition function of a Markov process on the measurable space (Z, \mathcal{Z}) . For any probability measure $\lambda \in \mathcal{P}(Z)$ and $A \in \mathcal{Z}$ we can define $T^*\lambda$ by

$$(T^*\lambda)(A) = \int P_z(z, A)\lambda(dz)$$

T^* is called the adjoint Markov operator and we can interpret $(T^*\lambda)(A)$ as the probability that the shock next period will be in the set A if the current realization is drawn from the distribution λ .

We call $\mu_z \in \mathcal{P}(Z)$ an *invariant distribution* for the transition function P_z if $T^*\mu_z = \mu_z$.

Remark. Intuitively, an invariant distribution leads to a situation in which the measure over states is constant over time, because in every period current realizations are drawn from the same distribution μ_z .

In BHA-Models consumers are hit by idiosyncratic shocks which are governed by a Markov process with a unique invariant distribution. This allows us to model consumers which behave differently even though they are ex-ante homogeneous. Heterogeneous behavior is now the result of different shock histories. The following definition provides a formal mathematical characterization of BHA models and mostly follows the notation from Acemoglu and Jensen (2015).

Definition 2.3 (BHA-Model). We have an infinite-horizon, discrete-time economy populated by a continuum of agents $\mathcal{I} = [0, 1]$ endowed with the Lebesgue measure. Each agent $i \in [0, 1]$ is subject to uninsurable idiosyncratic shocks $z_{i,t} \in Z \subseteq \mathbb{R}^k$ that follow a Markov process with transition function P_z and have a unique invariant distribution μ_z . Let $X \subseteq \mathbb{R}^l$ be an endogenous state variable. Then the state of an agent can be summarized as $s_i = (x_i, z_i) \in X \times Z$. Also, let $Q_t \in \mathcal{Q} \subseteq \mathbb{R}^q$ be an aggregate state at time

t .⁵ Given initial conditions $s_{i,0} = (x_{i,0}, z_{i,0})$, the agent solves

$$\begin{aligned} \sup \quad & \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t R(x_{i,t}, x_{i,t+1}, z_{i,t}) \right\} \\ \text{subject to} \quad & x_{i,t+1} \in \Gamma(x_{i,t}, z_{i,t}; Q_t), \quad t = 0, 1, 2, \dots \end{aligned} \quad (2.5)$$

where $\beta \in (0, 1)$ is the discount factor, $\Gamma : X \times Z \rightarrow 2^X$ the constraint correspondence and $R : X^2 \times Z \rightarrow \mathbb{R}$ the one-period return function.

Example 2.1. This definition might seem very abstract, but the notation becomes clear when translating it to the economy from Aiyagari (1994) described above. Here, the endogenous variable and the exogenous shock are one-dimensional, i.e. $k = l = 1$. The endogenous state variable (x) are asset holdings, a , which are determined by individual decisions. The exogenous state variable (z) is stochastic labor supply, h , which cannot be influenced by the consumer and is taken as given every period. Hence, the state of an agent consists of current asset holdings and the realization of the exogenous labor supply shock, which is the pair (a, h) . The correspondence constraint is the admissible set of the budget constrained defined in (2.2). To see this, solve for consumption, $c_{i,t}$, in terms of savings, $a_{i,t+1}$:

$$c_{i,t} = wh_{i,t} + (1 + r)a_{i,t} - a_{i,t+1},$$

and since we restrict the analysis to positive consumption levels, we get the constraint:

$$\begin{aligned} \Gamma(a_{i,t}, h_{i,t}; Q_t) = \left\{ a_{i,t+1} \in [-b, \infty) : a_{i,t+1} \leq wh_{i,t} + (1 + r)a_{i,t}, \right. \\ \left. w = w(Q_t), r = r(Q_t) \right\}, \end{aligned}$$

where we have signified that the wage rate and the interest rate will depend on the market aggregates (in this example, the profit maximization of the firm). With the same argument we can define the one-period return function when individuals only care about consumption by

$$R(a_{i,t}, a_{i,t+1}, h_{i,t}) = u\left(wh_{i,t} + (1 + r)a_{i,t} - a_{i,t+1}\right),$$

with $u(\cdot)$ being the instantaneous utility from consumption. Finally, the market aggregates are given by the capital and labor supply, $Q_t = (K_t, L_t)$, and determine the wage rate and the interest rate. With competitive markets, prices of inputs will be equal to their marginal products. Hence, $r = \delta F(K_t, L_t)/\delta K_t$ and $w = \delta F(K_t, L_t)/\delta L_t$, with $F(\cdot, \cdot)$ being a continuous production function.

⁵For example, Q_t could entail aggregate capital demand of the production sector, total labor supply in the economy, government consumption, etc.

In general, the solution to the problem in (2.5) from the point of view of an agent is a sequence of optimal response functions for any state s_t , because the agent realizes that the optimal decision will depend on the realization of the state s_t at every point in time. Hence, at time $t = 0$ we start with a known realization of the current state $s_0 = (x_0, z_0)$, with x_0 being the initial endogenous state variable and z_0 the initial realization of the exogenous shocks.⁶ Knowing this initial state s_0 , the agent makes a decision about x_1 . However, the decision about x_2 will depend on z_1 which is a random variable. Hence, the agent will make *contingency plans* for the periods $t = 1, 2, \dots$, which map the possible realizations of z_t to actual decisions. These contingency plans might be very complicated, but they have to satisfy certain restrictions in order for the optimization problem to be well posed. This observation gives rise to the following definition of individual strategies:

Definition 2.4 (Strategies). A strategy $\mathbf{a} = (a_0, a_1, a_2, \dots)$ is a value $a_0 \in X$ and a sequence of measurable functions $a_t : Z^t \rightarrow X$ for $t \geq 1$. We say a strategy is *feasible* if it satisfies the restriction in (2.5), and it is *optimal* if it solves the problem in (2.5). An optimal strategy is denoted by \mathbf{a}^* . We will denote with $z^t = (z_0, z_1, \dots, z_t) \in Z^t$ the partial history of shocks up until period t and write $a_t(z^t)$ for the value x_{t+1} that is chosen in period t when z^t is observed.

Remark. Note that the optimal strategy for the problem in (2.5) cannot simply be constructed pointwise by only looking at the periods which are affected by the choice $x_{i,t+1}$. We have

$$R(x_{i,t}, x_{i,t+1}, z_{i,t}) + \beta R(x_{i,t+1}, x_{i,t+2}, z_{i,t+1}),$$

and the optimal $x_{i,t+1}$ for this expression depends on the future shock $z_{i,t+1}$.

With this notation of strategies, we also get a corresponding sequence of distributions over the endogenous state X . For each agent $i \in \mathcal{I}$, the function $a_{i,t}$ maps the random shocks z^t into possible choices for the endogenous variable x_{t+1} and thus, it yields a probability distribution $\mu_{a_{i,t}} \in \mathcal{P}(X)$ on X . In particular, for any subset $A \subseteq X$, we have

$$\mu_{a_{i,t}}(A) = \mathbb{P}\left(\{z^t \in Z^t : a_{i,t}(z^t) \in A\}\right), \quad (2.6)$$

that is the probability of choosing the value $x_{i,t+1} \in A$ based on the strategy \mathbf{a}_i .

It is clear that the solution of a BHA model consists of optimal strategies $(\mathbf{a}_i^*)_{i \in \mathcal{I}}$. However, such a solution will not exist in general. For example, if we allow any arbitrary

⁶In the case of the model above, such an initial state would be current asset holdings and initial labor endowment. For example, we could think about new workers entering the model with initial state $s_0 = (0, 1)$, meaning that they start their model life with 0 assets and the median labor endowment.

return function R , the maximization problem might be undefined. Hence, we need to impose some restrictions on the model components to ensure the existence of a unique solution to the individual optimization problem.

Assumption 2.1. The transition function P_z has the Feller property⁷, X and Z are compact, R is bounded and continuous, and Γ is continuous with nonempty and compact values.

Lemma 2.1. *With Assumption 2.1 the set of feasible strategies is nonempty for all initial conditions $s_{i,0}$ and for all $i \in \mathcal{I}$. In particular, an optimal strategy \mathbf{a}^* exists.*

The proof of this lemma can be found in appendix A. Note that Assumption 2.1 is not very restrictive, as it will usually be satisfied by simply imposing some natural conditions on the economic parameters. In Aiyagari (1994), for example, employing a continuous production function already results in a compact constraint correspondence. Moreover, the assumptions of compactness for the state variables X and Z come about naturally, since they represent asset holdings and labor supply, both of which should be chosen from compact sets. Hence, only the Feller property of the transition function P_z is a real restriction. However, it is still fulfilled by most of the stochastic processes currently used in economic modelling.

In Example 2.1 we can see that the feasible strategies of agents depends on the wage rate and the interest rate, both of which are determined by the aggregate state. However, the agents' strategies can also be aggregated by mapping random variables (the strategies of the agents) into real numbers.

Definition 2.5 (Aggregator). An aggregator is a continuous and increasing function \mathcal{A} that maps the agents' strategies at time t into a real vector $Q_t \subseteq \mathcal{Q}$. The value

$$Q_t = \mathcal{A}\left((a_{i,t})_{i \in \mathcal{I}}\right) \quad (2.7)$$

is called the *market aggregate* at time t . Continuity is assumed with regards to the weak- $*$ -topology⁸ on the domain of \mathcal{A} , and \mathcal{A} is said to be increasing if $\mathcal{A}(x_1) \geq \mathcal{A}(x_2)$ for all random variables x_1 and x_2 such that x_1 stochastically dominates x_2 .

Remark. The aggregator cancels out individual uncertainty in the sense that it does not depend on any realization of the shock history, but only on the strategies themselves.

⁷A Markov operator is said to have the Feller property if it maps the set of bounded continuous functions into itself

⁸The weak- $*$ -topology is the coarsest topology under which all the linear functionals on the dual space are continuous.

Since the shocks are independent between individuals, with a law of large numbers for a continuum of agents as in Uhlig (1996) the transition function $P_z(z, dz)$ yields the deterministic fraction of the population that transitions from z to dz .

2.3 Sequential Equilibrium

In Figure 1a above we have already seen that the aggregate state will generally not be equal to the aggregation of individual actions for any possible strategy. There was only one value for capital demand such that the aggregation of individual strategies yields the same amount of savings. However, for a meaningful analysis the capital that is needed in production has to be equal to the available capital in the economy. Otherwise capital would simply be destroyed (in the case of excess supply), or created from nothing (in the case of excess demand). Hence, economists restrict their analyses to the equilibrium of a model. Following Acemoglu and Jensen (2015), the equilibrium of a BHA model as described in Definition 2.3 is given by a specific sequence of market aggregates and strategies:

Definition 2.6 (Sequential Equilibrium). For given initial conditions $(s_{i,0})_{i \in \mathcal{I}}$ a sequential equilibrium is a sequence of market aggregates and individual strategies, $\left\{ \mathbf{Q}^*, (\mathbf{a}_i^*)_{i \in \mathcal{I}} \right\}$, such that the following conditions hold:

- (i). Optimality: For each agent $i \in \mathcal{I}$, $\mathbf{a}_i^* = (a_{i,0}^*, a_{i,1}^*, a_{i,2}^*, \dots)$ solves the optimization problem in (2.5) given $\mathbf{Q}^* = (Q_0^*, Q_1^*, Q_2^*, \dots)$ and the initial conditions $(s_{i,0})_{i \in \mathcal{I}}$. This means:

$$\forall i \in \mathcal{I} : \mathbf{a}_i^* = \arg \sup_{\mathbf{a}_i \in \mathfrak{N}_i(s_{i,0}; \mathbf{Q}^*)} R(x_{i,0}, a_{i,0}(z_{i,0}), z_{i,0}) + \mathbb{E}_0 \left\{ \sum_{t=1}^{\infty} \beta^t R(a_{i,t-1}(z_i^{t-1}), a_{i,t}(z_i^t), z_{i,t}) \right\},$$

with $\mathfrak{N}_i(s_{i,0}; \mathbf{Q}^*)$ being the set of feasible strategies for agent i starting from $s_{i,0}$ and given \mathbf{Q}^* .

- (ii). Market Clearing: $\forall t \in \mathbb{N}_0 : Q_t^* = \mathcal{A} \left((a_{i,t}^*)_{i \in \mathcal{I}} \right)$, meaning that market aggregates are determined by individual actions.

The intuition of point (i) is straightforward: it simply states that in equilibrium all agents follow their optimal strategies. Point (ii) states that the aggregation of these individual strategies should be equal to the market aggregates that induce these strategies in the first place. In the example economy described above, this condition means that the

interest rate induced by the capital demand of the firm should lead to savings decisions of the households that, when aggregated, exactly satisfy this demand. In other words, the condition implies that there is no excess demand or supply of capital, meaning that the capital market is *cleared*.

The existence of an equilibrium in the market heavily depends on the characteristics of the market aggregation. Specifically, for an equilibrium to exist as defined earlier, it is essential that the aggregator \mathcal{A} maps all permissible subsets to convex images. If this requirement is not met, there may not be an equilibrium. Rather than mandating a particular aggregator with these desired features, we adopt an assumption proposed by Acemoglu and Jensen (2015) regarding the agents' problem in cases where the convexifying characteristic is absent.

Assumption 2.2. At least one of the following two conditions holds:

- (i). The aggregation function \mathcal{A} is convexifying; that is, for any subset B of the set of joint strategies such that $\mathcal{A}(b)$ is well defined for all $b \in B$, the image $\mathcal{A}(B) \subseteq \mathcal{Q}$ is convex.
- (ii). X is convex, and given any choice of z_i and Q , the return function $R(x_i, y_i, z_i)$ is concave in (x_i, y_i) and $\Gamma(x_i, y_i, z_i; Q)$ has a convex graph for each agent i .

Theorem 2.2 (Existence of Equilibrium). *Under Assumptions 2.1 and 2.2 there exists an equilibrium for any choice of initial conditions $(s_{i,0})_{i \in \mathcal{I}}$.*

Proof. For agent i let $\aleph_i(s_i, \mathbf{Q})$ denote the set of feasible strategies (which are infinite sequences of random variables) and $\gamma_i(\mathbf{Q}) \subseteq \aleph_i(s_i, \mathbf{Q})$ the set of optimal strategies given the sequence of aggregates $\mathbf{Q} \in \mathcal{Q}^{\mathbb{N}}$. With Assumption 2.1 and Lemma 2.1, $\gamma : \mathcal{Q}^{\mathbb{N}} \rightarrow 2^{\aleph}$ will be non-empty valued and upper hemicontinuous. Consider the upper hemicontinuous correspondence $\mathcal{A}(\mathbf{Q}) = \{\mathcal{A}((\mathbf{a}_i)_{i \in \mathcal{I}}) : \mathbf{a}_i \in \gamma_i(\mathbf{Q}) \text{ for } i \in \mathcal{I}\}$. If we can apply the Kakutani-Glicksberg-Fan fixed-point theorem for infinite-dimensional locally convex topological vector spaces (see Theorem B15 in Appendix B), the existence of a \mathbf{Q}^* as in Definition 2.6 follows immediately.

We can see that $\mathcal{A}(\mathbf{Q})$ will be non-empty and upper hemicontinuous. Furthermore, with Assumption 2.2 $\mathcal{A}(\mathbf{Q})$ will also be compact and convex valued for all \mathbf{Q} . Consequently, $\mathcal{A} : \mathcal{Q}^{\mathbb{N}} \rightarrow 2^{\mathcal{Q}^{\mathbb{N}}}$ is a Kakutani map and $\mathcal{Q}^{\mathbb{N}}$ with the supremum norm $\|\mathbf{Q}\|_{\infty} = \sup_t |Q_t|$ is a convex and compact topological space. Thus, the Kakutani-Glicksberg-Fan theorem applies and \mathcal{A} has a fixed point, i.e.:

$$\exists \mathbf{Q}^* : \mathbf{Q}^* \in \mathcal{A}(\mathbf{Q}^*).$$

This fixed-point \mathbf{Q}^* is a sequence of market aggregates with associated strategies $(\mathbf{a}_i^* \in \gamma_i(\mathbf{Q}^*))_{i \in \mathcal{I}}$ that satisfies Definition 2.6. Hence, $\{\mathbf{Q}^*, (\mathbf{a}_i^*)_{i \in \mathcal{I}}\}$ is an equilibrium outcome. \square

2.4 The Principle of Optimality

In a sequential equilibrium as discussed above, the equilibrium outcome may vary between each period, which can be problematic in economic analyses. As a result, economists often focus on stationary equilibria, where the market aggregate remains constant across periods. In other words, we seek to identify equilibria that correspond to a constant sequence of aggregate states $\mathbf{Q} = \{Q, Q, \dots\}$. By doing so, economists can better analyze the long-term behavior of markets and make more accurate predictions about future outcomes.

Before extending our discussion to these special kind of equilibrium, however, it will prove convenient to first introduce the notion of value functions and policy correspondences. To do so, let's look again at the individual's optimization problem:

$$\begin{aligned} \sup \quad & \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t R(x_{i,t}, x_{i,t+1}, z_{i,t}) \right\} \\ \text{subject to} \quad & x_{i,t+1} \in \Gamma(x_{i,t}, z_{i,t}; Q_t), \quad t = 0, 1, 2, \dots \end{aligned} \quad (2.8)$$

Now, since we are interested in stationary outcomes, we will assume that \mathbf{Q} is constant and fixed at a given value Q . Let $\aleph_i(s_{i,0}; Q)$ denote the set of feasible strategies for agent i with initial conditions $s_{i,0} = (x_{i,0}, z_{i,0})$ and the given market aggregate Q . As shown above, under Assumptions 2.1 and 2.2 this problem has a solution. We now write this solution as a function $v^* : X \times Z \rightarrow \mathbb{R}$, defined as

$$\begin{aligned} v^*(x_{i,0}, z_{i,0}; Q) = & \sup_{\mathbf{a}_i \in \aleph_i(s_{i,0}; Q)} R(x_{i,0}, a_{i,0}(z_{i,0}), z_{i,0}) + \\ & \mathbb{E}_0 \left\{ \sum_{t=1}^{\infty} \beta^t R\left(a_{i,t-1}(z_i^{t-1}), a_{i,t}(z_i^t), z_{i,t}\right) \right\} \end{aligned} \quad (2.9)$$

We call v^* the *value function* corresponding to the dynamic optimization problem in (2.8). The computation of this function v^* might not be straightforward and it would be convenient, if we could rewrite the problem in a functional relationship. Such a functional relationship could look as follows:

$$v(x, z; Q) = \sup_{y \in \Gamma(x, z; Q)} \left\{ R(x, y, z) + \beta \mathbb{E} [v(y, z'; Q)] \right\}, \quad (2.10)$$

where z is the current value of the exogenous shock, while z' is the respective value next

period. The expectation is formed with respect to the realizations of z' . If there exists a function v satisfying the functional relationship in (2.10), then we can also formulate the associated policy correspondence:

$$G(x, z; Q) = \left\{ y \in \Gamma(x, z; Q) : \right. \\ \left. v(x, z; Q) = R(x, y, z) + \beta \mathbb{E}[v(y, z'; Q)] \right\} \quad (2.11)$$

In the deterministic case (if the process z_t is deterministic), it can easily be shown that if the supremum function v^* in (2.9) is well-defined, it also satisfies the functional relationship in (2.10). Conversely, the solution to (2.10) is the supremum function. However, in a stochastic setting, the function v^* may not be measurable, making the expectation operation in the functional equation undefined. This issue is not just theoretical as demonstrated by an example constructed by Blackwell (1965) which shows that these measurability issues can arise even in simple settings. Therefore, we must limit our analysis to cases where certain criteria are met to get the following result as in Stokey et al. (1989b):

Theorem 2.3. *Let Assumptions 2.1 – 2.2 hold, and let v^* be defined as in (2.9). Let v be a measurable function satisfying the functional relationship given in (2.10), such that for all initial conditions $s_{i,0} = (x_{i,0}, z_{i,0}) \in X \times Z$ and all strategies $\mathbf{a}_i \in \aleph_i(s_{i,0}; Q)$, given the aggregate state Q , we have*

$$\lim_{t \rightarrow \infty} \mathbb{E} \left\{ \beta^t v(a_{i,t-1}(z_i^{t-1}), z_{i,t}; Q) \right\} = 0. \quad (2.12)$$

Let G be the associated policy correspondence defined in (2.11), and assume that G is non-empty and permits a measurable selection. Then $v = v^$ and any strategy \mathbf{a}_i generated by G attains the supremum in (2.9).*

Proof. For the sake of simplicity, we will omit the subscript i in the following proof, but note that we are still referring to the optimization problem of a particular agent.

Let's define a sequence of functions $U_n : \aleph(s_0; Q) \rightarrow \mathbb{R}$ by

$$U_0(\mathbf{a}, s_0; Q) = R(x_0, a_0, z_0) \\ U_n(\mathbf{a}, s_0; Q) = R(x_0, a_0, z_0) + \mathbb{E} \left\{ \sum_{t=1}^n \beta^t R(a_{t-1}(z^{t-1}), a_t(z^t), z_t) \right\}$$

Then we can define $U(\mathbf{a}, s_0; Q) = \lim_{n \rightarrow \infty} U_n(\mathbf{a}, s_0; Q)$ and have $v^* = \sup_{\mathbf{a} \in \aleph} U(\mathbf{a}, s_0; Q)$.

Under Assumptions 2.1 and 2.2, v^* is well-defined and the unique function satisfying:

$$v^*(s_0; Q) \geq U(\mathbf{a}, s_0; Q) \quad \text{for all } \mathbf{a} \in \aleph(s_0; Q) \quad (2.13)$$

$$v^*(s_0; Q) = \lim_{k \rightarrow \infty} U(\mathbf{a}^k, s_0; Q) \quad \text{for some } \{\mathbf{a}^k\}_{k=1}^{\infty} \in \aleph(s_0; Q) \quad (2.14)$$

Suppose v is a function satisfying the assumptions stated in Theorem 2.3 and the functional relationship (2.10). If we can show that v also satisfies conditions (2.13) and (2.14), then it follows that $v = v^*$.

Step 1: For any $\mathbf{a} \in \aleph(s_0; Q)$ we have

$$\begin{aligned} v(s_0; Q) &= \sup_{y \in \Gamma(s_0; Q)} \left\{ R(x_0, y, z_0) + \beta \mathbb{E}[v(y, z_1; Q)] \right\} \\ &\geq R(x_0, a_0, z_0) + \beta \mathbb{E}[v(a_0, z_1; Q)] \\ &= U_0(\mathbf{a}, s_0; Q) + \beta \mathbb{E}[v(a_0, z_1; Q)] \\ &= U_0(\mathbf{a}, s_0; Q) + \beta \mathbb{E} \left[\sup_{y \in \Gamma(a_0, z_1; Q)} \left\{ R(a_0, y, z_1) + \beta \mathbb{E}[v(y, z_2; Q)] \right\} \right] \\ &\geq U_0(\mathbf{a}, s_0; Q) + \beta \mathbb{E} \left[R(a_0, a_1(z^1), z_1) + \beta \mathbb{E}[v(a_1(z^1), z_2; Q)] \right] \\ &= U_1(\mathbf{a}, s_0; Q) + \beta^2 \mathbb{E}[v(a_1(z^1), z_2; Q)], \end{aligned}$$

where we have simply used the definitions of U_0 and U_1 and the fact that v satisfies the functional relationship in (2.10). By induction, for $n \geq 1$ we get:

$$v(s_0; Q) \geq U_n(\mathbf{a}, s_0) + \beta^{n+1} \mathbb{E}[v(a_n(z^n), z_{n+1}; Q)].$$

Using (2.12) yields

$$v(s_0, Q) \geq \lim_{n \rightarrow \infty} U_n(\mathbf{a}, s_0; Q) + 0 = U(\mathbf{a}, s_0; Q),$$

and since $\mathbf{a} \in \aleph(s_0)$ was arbitrary, the function v satisfies (2.13).

Step 2: Since G permits a measurable selection and is non-empty, there exists a sequence $\mathbf{g} = (g_0, g_1, \dots)$ of measurable selections from G . We define

$$\begin{aligned} a_0^* &= g_0(s_0) \\ a_t^*(z^t) &= g_t(a_{t-1}^*(z^{t-1}), z_t), \quad z^t \in Z^t, t = 1, 2, \dots \end{aligned}$$

Since the sequence $\mathbf{a}^* = (a_0^*, a_1^*, \dots)$ is generated by a composition of measurable functions, \mathbf{a}^* is a strategy. Furthermore, since $g_t(s) \in G(s; Q) \subseteq \Gamma(s; Q)$, \mathbf{a}^* is feasible and due to the definition of G we can repeat the argument in step 1 with equality in every line. Hence, the function v satisfies (2.14) with $\mathbf{a}^k = \mathbf{a}^*$ for all k .

Together, step 1 and step 2 imply $v(s_0) = v^*(s_0)$ and since s_0 was arbitrary, $v = v^*$. □

This theorem tells us, that v is a solution to the individual's maximization problem if v solves the functional relationship in (2.10). With this result we have a way to write a sequential problem as in (2.8) as a functional relationship in which we can find the unknown function v in order to solve the optimization. Note, however, that Theorem 2.3 only states that any strategy generated by the policy correspondence G will attain the supremum and thus be optimal. Nonetheless, not every optimal strategy must necessarily be induced by G .

Definition 2.7 (Policy Function). For any given state $s \in S$ and a given aggregate state Q , we call the measurable selection $g(s) \in G(s; Q)$ the *policy function* for an agent in state s .

Remark. The policy function provides the optimal decision for an agent whose current state is given by $s \in S$. Since the state space is the same for all agents ($S = X \times Z$ for all $i \in \mathcal{I}$), the value function and its associated policy function are also the same for all agents. Hence, if we know the distribution of agents over states, the policy function will yield a distribution over endogenous decisions.

With this notion of value functions and policy functions we are able to detach the solution of the optimization problem from a specific sequence of periods. Instead, we can characterize the optimal decisions of agents for any given current state, independent of the specific history that led to this state. Therefore, to solve the sequential problem in (2.8) we can write it *recursively* as

$$v(x, z; Q) = \sup_{y \in \Gamma(x, z; Q)} \left\{ R(x, y, z) + \beta \mathbb{E} \left[v(y, z'; Q) \right] \right\},$$

where we adopt the common notation that next period's values are denoted with a prime, $z \rightarrow z'$. Theorem 2.3 tells us that the unique solution v^* to this functional equation will solve the dynamic optimization problem, and the optimal strategies will be generated by the associated policy correspondence as defined in (2.11). This result constitutes the so-called *principle of optimality* as described by Bellman (1957).

2.5 Stationary Equilibria

For the evaluation of policies and the exploration of economic mechanisms it is often inconvenient to look at a sequential equilibrium as in Definition 2.6. With strategies and market aggregates being dependent on the periods t , the effects of economic channels might only be present during certain points in time or while converging to infinity. Hence, economists usually focus on so-called stationary equilibria, in which the market aggregates do not change between periods. These equilibria are particularly useful when studying counterfactual economies or comparing the effects of different policies.

Before providing the definition of a stationary equilibrium, we need to establish the notion of stationary strategies. Remember that the measurable functions $a_{i,t}$ of a strategy \mathbf{a}_i give rise to a probability distribution $\mu_{a_{i,t}}$ over X as in (2.6).

Definition 2.8 (Stationary Strategies). A strategy \mathbf{a}_i is called *stationary*, if for any two periods $t_1, t_2 \in \mathbb{N}$, a_{i,t_1} and a_{i,t_2} yield the same distribution, i.e. $\mu_{a_{i,t_1}} = \mu_{a_{i,t_2}}$ on $(X, \mathcal{B}(X))$. We say, \mathbf{a}_i generates the stationary probability distribution $\mu_{\mathbf{a}_i}$.

Definition 2.9 (Stationary Equilibrium). A stationary equilibrium is a constant sequence of aggregate states and a sequence of stationary strategies, $\{\mathbf{Q}^*, (\mathbf{a}_i^*)_{i \in \mathcal{I}}\}$, such that the following conditions hold:

- (i). Optimality: For each agent $i \in \mathcal{I}$, the stationary strategy $\mathbf{a}_i^* = (a_{i,1}^*, a_{i,2}^*, a_{i,3}^*, \dots)$ with stationary distribution $\mu_{\mathbf{a}_i^*}$ solves the optimization problem in (2.5) given the constant sequence of aggregate states $Q^* = (Q^*, Q^*, Q^*, \dots)$ and the randomly drawn initial conditions $(x_{i,0}, z_{i,0}) \sim \mu_{\mathbf{a}_i^*} \times \mu_z$.
- (ii). Market Clearing: $\forall t \in \mathbb{N}_0 : Q^* = \mathcal{A}\left((a_{i,t}^*)_{i \in \mathcal{I}}\right)$, meaning that the aggregate state is equal to the market aggregate of individual actions.

Remark. Optimality in the above definition refers to a solution of the problem in (2.5). Hence, given the constant sequence \mathbf{Q}^* , \mathbf{a}_i^* is optimal among all strategies, not only stationary strategies. Also, when drawing the initial conditions, $x_{i,0}$ and $z_{i,0}$ are assumed to be independent. Lastly, note that the market clearing condition is the same as for a sequential equilibrium, with the only exception that the market aggregate of individual actions is the same for every period t . In particular, every stationary equilibrium is also a sequential equilibrium with constant aggregate states.

The intuition behind this definition is that in a stationary equilibrium the actions of individuals lead to the same aggregate economic outcomes every period, regardless of the specific shocks that may occur. While individual actions may still vary from one

period to another in response to different shocks, the distribution of possible actions remains constant over time. This means that, on average, the fraction of the population transitioning into and out of specific states always balances out every period. This can be explained using a law of large numbers for a continuum of agents as in Uhlig (1996).

Example 2.2. Consider again the Aiyagari (1994) economy from Section 2.1. In Figure 1a we have seen that for different interest rates the aggregate savings decisions of the agents might diverge from the capital demand of firms. However, at an interest rate of roughly 1.2% both aggregates coincide. A stationary equilibrium now describes a situation in which the interest rate is equal to 1.2% in every period and thus, every period the market aggregate and the aggregation of individual savings decisions will coincide and be constant over time.

To ensure the existence of such a stationary distribution, we first introduce the notion of a lattice and supermodular function and then impose additional restrictions on our optimization problem.

Definition 2.10. Let A and B be partially ordered sets, $\Gamma : A \times B \rightarrow 2^A$ a correspondence, and $f : A \times B \rightarrow \mathbb{R}$ a mapping. Denote with $a_1 \vee a_2$ the supremum and with $a_1 \wedge a_2$ the infimum of the two-element subset $\{a_1, a_2\}$. We say

- A is a **lattice** if for any two-element subset $\{a_1, a_2\} \subseteq A$ both $a_1 \vee a_2 \in A$ and $a_1 \wedge a_2 \in A$.
- for fixed b , the graph of Γ is a **sub-lattice** of $A \times A$ if $c_1 \in \Gamma(a_1)$ and $c_2 \in \Gamma(a_2)$ implies that $c_1 \vee c_2 \in \Gamma(a_1 \vee a_2)$ and $c_1 \wedge c_2 \in \Gamma(a_1 \wedge a_2)$.
- the function f is **supermodular** in (a, b) if

$$f(a_1 \vee a_2, b_1 \vee b_2) + f(a_1 \wedge a_2, b_1 \wedge b_2) \geq f(a_1, b_1) + f(a_2, b_2)$$

Assumption 2.3. The set of endogenous variables X is a lattice. Also, given any choice of z_i , the one-period return function $R(x_i, y_i, z_i)$ is supermodular in (x_i, y_i) and the graph $\Gamma(x_i, z_i; Q)$ is a sublattice of $X \times X$ for any choice of z_i and Q .

The restrictions imposed in Assumption 2.3 are again not very strong as they are commonly satisfied in typical economic applications. For example, the graph of Γ will be a sub-lattice if Γ is ascending in x , meaning that for $x_2 \geq x_1$, $y_1 \in \Gamma(x_1)$ and $y_2 \in \Gamma(x_2)$ implies that $y_1 \vee y_2 \in \Gamma(x_2)$ and $y_1 \wedge y_2 \in \Gamma(x_1)$. Intuitively, this can be interpreted in the sense that a higher endogenous state today allows a higher endogenous state next period, which is a natural requirement for economic actions. Similarly, in most economic

settings, supermodularity of the return function is already fulfilled by using a concave utility function.⁹

With Assumption 2.3 we get additional structure for our policy correspondence which will translate to the operator that we are going to use to find an invariant distribution.

Lemma 2.4. *With Assumptions 2.1 and 2.3 the policy correspondence $G : X \times Z \rightarrow 2^X$ as defined in (2.11) will be ascending in x for any fixed value of z .*

Intuitively, Lemma 2.4 tells us that the decision about next period's endogenous state is increasing in the current value. For example, higher asset holdings today would induce higher savings tomorrow. The proof of the lemma can be found in Appendix A.

Now, for any agent i and a fixed aggregate state Q , the policy correspondence G_i obtained from the optimal strategy is

$$G_i(x_i, z_i; Q) = \left\{ y_i \in \Gamma(x_i, z_i; Q) : v_i^*(x_i, z_i; Q) = R(x_i, y_i, z_i) + \beta \mathbb{E} \left[v_i^*(y_i, z'_i; Q) \right] \right\},$$

where v^* denotes the value function solving the functional relationship in (2.10). With Lemma 2.4 we know that this correspondence will be ascending in x . We denote with g_i a measurable selection from G_i (the policy function associated with the optimal strategy). With μ_z being the invariant distribution of the Markov process for the exogenous shock we can define a transition function for the endogenous state, $P_{g_i, Q} : X \times \mathcal{B}(X) \rightarrow [0, 1]$, by

$$P_{g_i, Q}(x_{i,t}, A) \equiv \mu_z(\{z_{i,t} \in Z : g_i(x_{i,t}, z_{i,t}; Q) \in A\}), \quad (2.15)$$

where $A \in \mathcal{B}(X)$. This transition function tells us the probability of choosing next period's value from the set A if the current value is $x_{i,t}$. However, the current value has already been decided last period depending on the realization of the exogenous shock. Hence, we would like to find a formulation that provides us with a probability measure $\mu_{t+1} \in \mathcal{P}(X)$ given that current period's values have been chosen according to the probability measure μ_t .

Definition 2.11 (Adjoint Markov Operator). For any agent $i \in \mathcal{I}$, let $P_{g_i, Q}$ denote the transition function for the endogenous state X based on the optimal decision rule g_i and the aggregate state Q as defined in (2.15). For any probability measure $\mu \in \mathcal{P}(X)$ define the operator $T_{g_i, Q}^*$ by

$$T_{g_i, Q}^* \mu = \int P_{g_i, Q}(x, \cdot) \mu(dx). \quad (2.16)$$

⁹For example, with $R(x, y, z) = u(wz + (1+r)x - y)$ as in the Aiyagari economy in Example 2.1, we get $D_{xy}^2 u = -(1+r)u'' \geq 0$ if $u'' \leq 0$, and since $D_{xy}^2 \geq 0$ implies supermodularity, a concave function u satisfies Assumption 2.3.

We call $T_{g_i, Q}^*$ the adjoint Markov operator for the associated transition function $P_{g_i, Q}$.

We can interpret $T_{g_i, Q}^* \mu(A)$ as the probability that the endogenous variable will lie in the set A next period given that the current state is drawn from the probability distribution μ . In other words, for any probability measure μ_t over the endogenous state at time t , $\mu_{t+1} = T_{g_i, Q}^* \mu_t$ is the probability measure over the endogenous state in period $t + 1$. Hence, for any agent i the measure μ_i^* is associated with a stationary strategy for a given aggregate state Q if and only if μ_i^* is a fixed point of $T_{g_i, Q}^*$, i.e. $\mu_i^* = T_{g_i, Q}^* \mu_i^*$. If such a stationary distribution exists for all agents $i \in \mathcal{I}$, we are one step closer to proving the existence of a stationary equilibrium.

Definition 2.12 (First-order stochastic dominance). For any pair of distributions μ_1 and μ_2 in $\mathcal{P}(X)$, we write $\mu_1 \succeq \mu_2$ if for every increasing, measurable, and bounded function $f : X \rightarrow \mathbb{R}$, we have

$$\int f(x) \mu_1(dx) \geq \int f(x) \mu_2(dx).$$

We call \succeq the first-order stochastic dominance order.

Lemma 2.5. *Suppose Assumptions 2.1 - 2.3 hold. For a given aggregate state Q and a measurable selection g from the optimal policy correspondence G , the adjoint Markov operator as defined in (2.16) will be increasing in the sense that for two measures $\mu_2 \succeq \mu_1$ we have $T_{g_i, Q}^* \mu_2 \succeq T_{g_i, Q}^* \mu_1$.*

Again, the proof of this result is provided in Appendix A. Intuitively, Lemma 2.5 means that if the current state is drawn from a distribution that puts more weight on higher asset holdings, the probability distribution for next period's values will also put more weight on higher asset holdings. This result is closely connected to the ascending policy correspondence, but generalized for a randomly drawn endogenous state.

Theorem 2.6 (Existence of Stationary Equilibrium). *Let Assumptions 2.1- 2.3 be satisfied. Then there exists a stationary equilibrium.*

Proof. For any agent $i \in \mathcal{I}$, $T_{g_i, Q}^*$ is an increasing map when equipping $\mathcal{P}(X)$ with the first-order stochastic dominance order \succeq (see Lemma 2.5). Since X is assumed to be compact, we have $m \equiv \inf(X) \in X$. Let δ_m be the measure that assigns probability 1 to the point set $\{m\}$. Then $\mu \succeq \delta_m$ for all $\mu \in \mathcal{P}(X)$, in particular $T_{g_i, Q}^* \delta_m \succeq \delta_m$, and from Theorem 1 in Hopenhayn and Prescott (1992)¹⁰ it follows that $T_{g_i, Q}^*$ has a fixed-point μ_i^* .

Let \mathcal{G}_i denote the set of measurable selections from G_i . Then, the adjoint Markov correspondence is given by

$$T_{i, Q}^* \mu = \{T_{g_i, Q}^* \mu\}_{g_i \in \mathcal{G}_i},$$

¹⁰See Theorem B11 in Appendix B.

and we define the invariant distribution correspondence $F_i : \mathcal{Q} \rightarrow 2^{\mathcal{P}(X)}$ as $F_i(Q) = \{\mu \in \mathcal{P}(X) : \mu \in T_{i,Q}^* \mu\}$. Note that $F_i(Q)$ will be non-empty valued and upper hemicontinuous. Define the joint correspondence over all agents as

$$F = (F_i)_{i \in \mathcal{I}} : \mathcal{Q} \rightarrow 2^{\left(\mathcal{P}(X)^{\mathcal{I}}\right)}.$$

Then, for any distribution $\mu \in F(Q)$ and for sets $A_i \in \mathcal{B}(X)$, we have

$$\mu \left(\prod_{i \in \mathcal{I}} A_i \right) = \prod_{i \in \mathcal{I}} \mu_i(A_i),$$

with $\mu_i \in F_i(Q)$. Denote the random variable $id : X^{\mathcal{I}} \rightarrow X^{\mathcal{I}}$ on the probability space $(X^{\mathcal{I}}, \mathcal{B}(X^{\mathcal{I}}), \mu)$ by $\hat{\mu}$. Given the aggregator \mathcal{A} , define a mapping $\tilde{\mathcal{A}}$ by the convention that $\tilde{\mathcal{A}}(\mu) = \mathcal{A}(\hat{\mu})$ and consider

$$\hat{\mathcal{A}}(Q) = \{\tilde{\mathcal{A}}(\mu) \in \mathbb{R}^g : \mu \in F(Q) \text{ for all } i\}$$

It is clear that Q^* is the aggregate state of a stationary equilibrium if and only if $Q^* \in \hat{\mathcal{A}}(Q^*)$. Since $\hat{\mathcal{A}}$ is an upper hemicontinuous and convex-valued correspondence that maps a compact and convex subset of \mathbb{R}^g into itself, we can apply the Kakutani fixed-point theorem as in the proof of Theorem 2.2. This shows the existence of an aggregate state $Q^* \in \hat{\mathcal{A}}(Q^*)$ and hence, the existence of a stationary equilibrium. □

3 Computational Solution Methods

With the results in Section 2 we now know under which conditions we can study a general equilibrium of a BHA economy. For economic applications, however, we also need to find the solutions. Since economists are usually only interested in stationary equilibria as discussed in Section 2.5, in the following we will focus on stationary solution methods only. With the principle of optimality, finding a stationary solution to the sequential optimization problem in Definition 2.3 is equivalent to solving for the unknown function v in the functional relationship (2.10). In the following, we will discuss the three main types of computational methods which can be used to find the value function v . Note that since we focus on stationary equilibria, we will always assume that the sequence of market aggregates, \mathbf{Q} , is constant and we will denote with $S = X \times Z$ the state space consisting of endogenous choices and exogenous shocks.

3.1 Guess and Verify

For simple problems, the solution to the functional relationship in (2.10) can sometimes be obtained with the "Guess and Verify" approach, meaning that we guess the functional form of the value function and then verify our guess by finding the parameters which solve the equation.

Example 3.1. Consider a simplified version of the Aiyagari model in Section 2.1, where we ignore any income or interest from savings, meaning that we are only concerned with optimally splitting some given resources between periods. The budget constraint then reads

$$a_{t+1} = a_t - c_t.$$

Also, per-period utility is given by $\log(c_t)$. Hence, the optimization problem is

$$\begin{aligned} & \max \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\} \\ & \text{subject to} \\ & a_{t+1} = a_t - c_t, \end{aligned}$$

Using the budget constraint we can substitute c_t and write the functional relationship for our value function as¹¹

$$v(a) = \max_{a' \in \Gamma} \log(a - a') + \beta v(a').$$

Now we make a guess on the value function and assume it takes the form $v(a) = A + B \cdot \log(a)$ for some values A and B . Inserting this guess into the Bellman equation yields:

$$v(a) = A + B \cdot \log(a) = \max_{a' \in \Gamma} \log(a - a') + \beta A + \beta B \log(a')$$

Taking first order conditions yields

$$a' = \frac{B\beta a}{1 + B\beta}$$

Next, re-write the Bellman equation in terms of the optimal a' :

$$A + B \log(a) = \log\left(a - \frac{B\beta a}{1 + B\beta}\right) + \beta A + \beta B \log\left(\frac{B\beta a}{1 + B\beta}\right)$$

¹¹Remember the convention that we denote next periods values with a prime, meaning that a' is the choice for next periods asset holdings.

We have

$$\log\left(a - \frac{B\beta a}{1 + B\beta}\right) = \log\left(\frac{a}{1 + B\beta}\right) = \log(a) - \log(1 + B\beta),$$

and

$$\beta B \log\left(\frac{B\beta a}{1 + B\beta}\right) = \beta B \log(a) + \beta B \log\left(\frac{B\beta}{1 + B\beta}\right).$$

Thus, we get:

$$\begin{aligned} A + B \log(a) &= \log(a) - \log(1 + \beta B) + \beta A + \beta B \log(a) + \beta B \log\left(\frac{B\beta}{1 + B\beta}\right) \\ &= (1 + \beta B) \log(a) - \log(1 + \beta B) + \beta A + \beta B \log\left(\frac{B\beta}{1 + B\beta}\right) \end{aligned}$$

Now, we can simply compare the coefficients and get

$$\begin{aligned} B &= \frac{1}{1 - \beta} \\ A &= (1 - \beta)^{-1} \log(1 - \beta) + \beta(1 - \beta)^{-2} \log(\beta), \end{aligned}$$

which are both constants as assumed above and we have verified our guess.

The example above shows one of the advantages of the guess and verify approach: If successful, this approach provides a correct analytical solution. However, it also highlights some significant disadvantages. First, we had to simplify the economic model and while the resulting solution might be analytically correct, it does not provide any meaningful economic insights. Also, we need an educated guess on the functional form of the value function v . Oftentimes this means that models are designed in a way that is likely to result in a value function whose functional form is already known, rather than concentrating on interesting or important economic channels. This introduces a serious constraint for economic analyses.

3.2 Value Function Iteration

Nowadays, the availability of high-quality micro-data has led to a trend towards increasingly complex models which are able to replicate several empirical moments. These models are too intricate to be solved analytically. Instead, numerical algorithms are used which provide an approximation of the true solution. One such method has proved very robust and is applicable to nearly every situation: value function iteration.

The idea of this method is straightforward, as we simply proceed by constructing a sequence of value functions and associated policy functions by iteratively applying the

same operation. Suppose we have a guess v_0 for the optimal solution v^* , with $v_0 \neq v^*$. Then we compute:

$$v_{new} = \sup_{y \in \Gamma(x, z; Q)} \left\{ R(x, y, z) + \beta \mathbb{E} [v_0(y, z'; Q)] \right\}.$$

Since v^* is the unique solution of the functional relationship in (2.10), $v_{new} \neq v_0$. But what can we say about the 'distance' to the optimal solution? If we could establish that $\|v_{new} - v^*\| \leq \|v_0 - v^*\|$, then we would have come closer to the true solution. The next theorem formally establishes this result.

Theorem 3.1. *Let Assumptions 2.1 and 2.2 hold and define the operator H by*

$$(Hv)(x, z; Q) = \sup_{y \in \Gamma(x, z; Q)} \left\{ R(x, y, z) + \beta \mathbb{E} [v(y, z'; Q)] \right\}. \quad (3.1)$$

Then $H : C(S) \rightarrow C(S)$, has a unique fix point v in $C(S)$ and for $v_0 \in C(S)$

$$\|H^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \text{ for } n \geq 1 \quad (3.2)$$

Moreover, the correspondence $G : S \rightarrow X$ defined by

$$G(x, z; Q) = \left\{ y \in \Gamma(x, z; Q) : v(x, z; Q) = R(x, y, z) + \beta \mathbb{E} [v(y, z'; Q)] \right\}$$

is non-empty, compact and upper hemicontinuous.

Proof. We follow the idea in Stokey et al. (1989b): Fix any $f \in C(S)$. Then the operator

$$(Mf)(x, z; Q) = \mathbb{E} \left[f(x, z'; Q) \mid z \right]$$

is clearly bounded, since $\|Mf\| \leq \|f\|$. But M is also continuous. To see this, choose a sequence $(x_n, z_n) \rightarrow (x, z)$. Then,

$$\begin{aligned} & |(Mf)(x, z; Q) - (Mf)(x_n, z_n; Q)| \\ & \leq |(Mf)(x, z; Q) - (Mf)(x, z_n; Q)| + |(Mf)(x, z_n; Q) - (Mf)(x_n, z_n; Q)| \\ & \leq |(Mf)(x, z; Q) - (Mf)(x, z_n; Q)| + \mathbb{E} \left[|f(x, z'; Q) - f(x_n, z'; Q)| \right] \end{aligned}$$

Now, Z is compact and with the feller property of the Markov process follows

$$\lim_{n \rightarrow \infty} |(Mf)(x, z; Q) - (Mf)(x, z_n; Q)| \rightarrow 0.$$

Also, since $X \times Z$ is compact and f continuous, f is uniformly continuous on $X \times Z$. Thus, for any $\varepsilon > 0$ there exists $N \geq 1$ such that for all $n > N$ and $z' \in Z$

$$|f(x, z'; Q) - f(x_n, z'; Q)| < \varepsilon$$

Together we can infer that $|(Mf)(x, z; Q) - (Mf)(x_n, z_n; Q)| \rightarrow 0$ and M is continuous.

Since R and Mv are bounded, Hv is also bounded. Moreover, since Γ is compact-valued and continuous, the Theorem of the Maximum (see Theorem B13 in Appendix B) tells us that Hv also has to be continuous, meaning $H : C(S) \rightarrow C(S)$.

Next, for $v_2 \geq v_1$ we clearly have $Hv_2 \geq Hv_1$, and for any constant function c $H(v + c) = Hv + \beta c$. Hence, H satisfies Blackwell's sufficient condition for a contraction (Theorem B12), and since $C(S)$ is a Banach space, the Contraction Mapping Theorem (Theorem B14) tells us that H has a unique fixed point $v \in C(S)$ and (3.2) holds. Lastly, the stated properties for G follow from the Theorem of the Maximum. □

With this result we have a straightforward way to obtain the optimal value function v^* by simply iteratively applying the operator defined in (3.2) to an initial guess v_0 :

$$v^* = \lim_{n \rightarrow \infty} H^n v_0$$

This iteration procedure can easily be implemented numerically, if we have a suitable method to apply this operation. The most commonly employed solution strategy is based on the *discretization* of the state space. To do so, we transform the state space $S = X \times Z$ into a discrete grid: Choose lower and upper bounds (\underline{x}, \bar{x}) and a number of grid points n_x for the endogenous variable X . Then we form a grid $\hat{X} = \{\underline{x}, \dots, \bar{x}\}$ such that $|\hat{X}| = n_x$.¹² With the same procedure we construct a grid \hat{Y} , $|\hat{Y}| = n_y$ for the set of possible choices, with $\hat{Y} \subseteq X$ (we need not restrict the choices to the same grid as the state variables). Finally, we do the same for the exogenous shocks z with n_z the number of grid points to get a grid $\hat{Z} = \{\underline{z}, \dots, \bar{z}\}$. To discretize the Markov transition function P_z of the exogenous shocks we use the same number of grid points n_{z_j} and a quadrature method such as the one described by Tauchen (1986).

Then, a fully discretized algorithm implementing value function iteration can be described as follows:

¹²Grids do not have to be constructed with equally spaced grid points. Better results can often be achieved by choosing the spacing of the grid points more carefully. For example, we might allow for more grid points within subsets that contain a high mass of agents, while making the grid more dispersed for choices which are likely to be obtained by only a negligible fraction of the population (e.g.: using more grid points around small asset holdings, while making the grid more dispersed for extreme wealth holdings).

Algorithm 3.1 (Discretized VFI).

Discretize the state space S and the transition function P_z .

Declare initial value v_0 (an array of dimension $n_x \times n_z$).

Declare iteration count $n = 0$.

Declare the tolerance level ε .

while $\|v_{n+1} - v_n\| > \varepsilon$ **do**

 Set $n = n + 1$.

 Set $v_{old} = v_{n-1}$.

for $x = 1, \dots, n_x$ **do**

for $z = 1, \dots, n_z$ **do**

 Calculate $\mathbb{E}[v_{old}|z]$

 Calculate $v_n(x, z) = \max_{y=1, \dots, n_y} R(x, y, z) + \beta \mathbb{E}[v_{old}(y, z'|z)]$

 Calculate $g_n(x, z) = \arg \max_{y=1, \dots, n_y} R(x, y, z) + \beta \mathbb{E}[v_n(y, z'|z)]$

end for

end for

end while

With the findings presented in above, this algorithm is expected to converge when the grid is fine enough (we will discuss this issue more closely in Section 4 below). However, this approach can be computationally expensive as optimization is required in every step and convergence tends to be slow. Although the process can be accelerated by using coarser discretized grids, this comes at the cost of reduced precision and may risk convergence issues.

3.3 Howard's Improvement Algorithm

As noted above, Algorithm 3.1 can be slow and computationally expensive. However, empirical observations in various applications have revealed that the optimal policy function often converges faster than the value function. This led to the development of an improved discretized algorithm that is usually less computationally expensive than simple value function iteration. The idea is to refrain from performing optimization in every step. Instead, we calculate the optimal policy rule for a given guess on the value function and then update the value function a couple of times with this policy rule. After a given number of iterations we use the new value function to update the policy rule by performing one optimization step and then go back to updating the value function for this new policy rule. Since the policy rule often converges much faster than the value function, this procedure

can drastically enhance computational performance. However, since we are combining optimization steps with 'naive' updates, we first need to establish convergence criteria.

Lemma 3.2. *Suppose Assumptions 2.1 and 2.3 hold. For a given policy function $g : X \times Z \rightarrow X$ we define the operator*

$$(H_g v)(x, z; Q) = R(x, g(x, z), z) + \beta \mathbb{E} \left[v(g(x, z), z'; Q) \right]. \quad (3.3)$$

Then $H_g : C(S) \rightarrow C(S)$ and there exists a unique fixed point $v_g \in C(S)$.

Proof. The proof follows the same arguments as the proof of Theorem 3.1. □

The operator defined in (3.3) calculates a new value function when using the policy function g as decision rule. Lemma 3.2 tells us that by iteratively applying H_g on an initial guess v_0 we will find a fixed-point v_g which will depend on the chosen policy function g . Note that the advantage of this approach is that we do not have to perform any optimization. However, in general the function v_g will not be equal to the optimal value v^* . Therefore, if we want to get an improvement on the initial guess v_0 , we need to choose the policy function with which to perform the operation with care. One idea is to take the policy rule corresponding to the guess v_0 , meaning that we perform one optimization step to find the optimal decision rule associated with v_0 and then use this decision rule for the operator H_g . Intuitively, using one optimization step should lead to some improvement in the value function. Then, after some steps of applying H_g , we could calculate a new decision rule based on the obtained value function and repeat the process. Formalizing this idea motivates the following theorem as in Ljungqvist and Sargent (2004):

Theorem 3.3. *Suppose Assumptions 2.1 and 2.3 are satisfied. Let H_g denote the operator as defined in (3.3) and $v_g = H_g v_g$. With H being the usual value function operator as in (3.1), find a new policy f such that*

$$H_h v_g = H v_g,$$

meaning that h is the optimal policy function when using v_g as the continuation value. Compute the fixed point v_h :

$$v_h = H_h v_h.$$

Then $v_h \geq v_g$. If g is not the optimal policy corresponding to v^ , then $v_h > v_g$ for at least one $x \in X$.*

Proof. We have

$$\begin{aligned}
H_h(v_g(x, z)) &= R(x, h(x, z), z) + \beta \mathbb{E} \left[v_g(h(x, z), z') \right] \\
&= H(v_g(x, z)) \\
&\geq R(x, g(x, z), z) + \beta \mathbb{E} \left[v_g(g(x, z), z') \right] \\
&= H_g(v_g(x, z)) \\
&= v_g(x, z).
\end{aligned}$$

Iteratively applying H_h and using the monotonicity of H_h yields

$$v_h = \lim_{n \rightarrow \infty} H_h^n(v_g) \geq v_g.$$

Assume $v_h(x) = v_g(x)$ for all $x \in X$. Then,

$$v_g = H_h(v_g) = H(v_g)$$

But if $v_g = H v_g$, then $v_g = v^*$ and v_g already satisfies the optimization problem. □

This theorem tells us that our idea from above is justified. We can find an improvement on the value function without having to use an optimization procedure by simply applying the operator as defined in (3.3) for a given policy function. This observation leads to a simple algorithm based on the work by Howard (1960), which mostly updates the value function using a fixed decision rule. Then, after a given number of iterations the updated value function can be used to find an improvement on the policy rule. This includes one optimization step. Overall, if the policy function converges faster than the value function, we will end up with an algorithm that uses less optimization steps than the algorithm for pure value function iteration. This should lead to a vastly improved performance, since it will in general require far less computing power. Using the same discretization technique as in Section 3.2, such an algorithm can be implemented as follows:

Algorithm 3.2 (Howard’s Improvement Algorithm).

Discretize the state space S and the transition function P_z .

Declare initial value v_0 (an array of dimension $n_x \times n_z$).

Declare iteration count $n = 0$.

Declare the tolerance level ε .

Declare number of policy improvements H .

```

while  $\|v_{n+1} - v_n\| > \varepsilon$  do
  Set  $n = n + 1$ .
  Set  $v_{old} = v_{n-1}$ .
  for  $x = 1, \dots, n_x$  do
    for  $z = 1, \dots, n_z$  do
      Calculate  $\mathbb{E}[v_{old}|z]$ 
      Calculate  $v_n(x, z) = \max_{y=1, \dots, n_y} R(x, y, z) + \beta \mathbb{E}[v_{old}(y, z'|z)]$ 
      Calculate  $g(x, z) = \arg \max_{y=1, \dots, n_y} R(x, y, z) + \beta \mathbb{E}[v_{old}(y, z'|z)]$ 
    end for
  end for
  for  $i = 1, \dots, n_H$  do
     $v_{old} = v_n$ 
    for  $x = 1, \dots, n_x$  do
      for  $z = 1, \dots, n_z$  do
        Calculate  $v_n(x, z) = R(x, g(x, z), z) + \beta \mathbb{E}[v_{old}(g(x, z), z'|z)]$ 
      end for
    end for
  end for
end while
for  $x = 1, \dots, n_x$  do
  for  $z = 1, \dots, n_z$  do
    Calculate  $g_n(x, z) = \arg \max_{y=1, \dots, n_y} R(x, y, z) + \beta \mathbb{E}[v_n(y, z'|z)]$ 
  end for
end for

```

Figure 2 graphically illustrates the difference between using an algorithm based on pure value function iteration (Algorithm 3.1) and an algorithm using Howard’s improvement strategy (Algorithm 3.2) when the number of iterations without policy updates, n_H , is chosen to be 100 and 200. The model which was solved for this comparison was the same

¹³As a stopping criterion $\varepsilon = 10^{-5}$ was used.

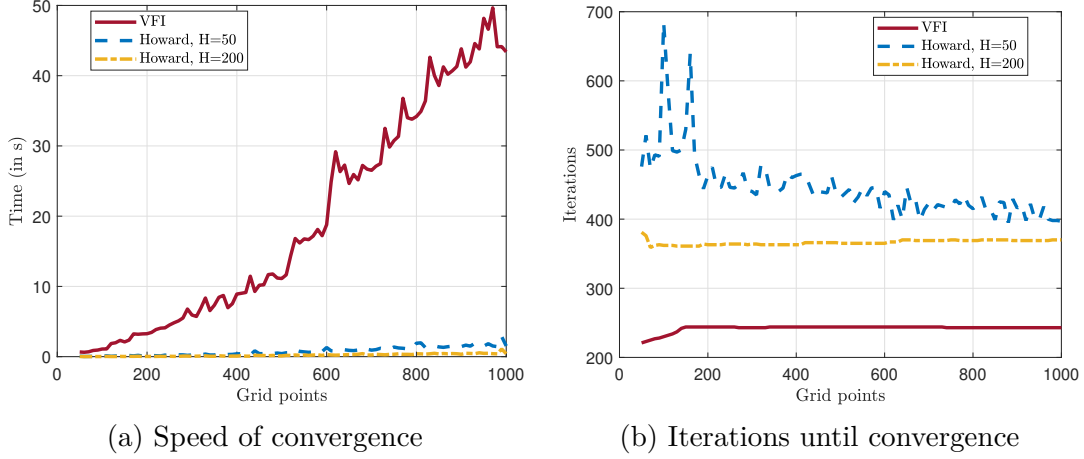


Figure 2: Comparison of convergence properties for pure value function iteration (VFI) and Howard's Improvement Algorithm.¹³

economy as the one discussed in Section 2.1 and with the parameterization described in Appendix C.¹⁴ We can immediately see how much faster Howard's Improvement Algorithm is, especially as the number of grid points increases. While the time consumption of pure value function iteration increases exponentially with the grid size, Howard's Improvement Algorithm proves to be considerably more efficient. Interestingly, this advantage in time consumption comes despite a substantially higher number of iterations, as evident from Figure 2b. This is because Howard's Improvement Algorithm requires more updates of the value function while keeping the policy rule fixed. However, the algorithm only needs to perform an optimization after every n_H -th step, whereas pure value function iteration constantly optimizes the decision rules. Since maximization operators are computationally expensive, this leads to the relatively poor performance of pure value function iteration. Lastly, we can see that regarding time consumption the difference between the choices for n_H seem to be negligible. However, choosing a lower number leads to considerably more fluctuations in the iterations needed to find a solution. This is because with a lower n_H we perform more optimizing updates on the value function.

3.4 Equilibrium Aggregates

In order to determine an equilibrium of the economy we not only need to solve for the individual decisions, but also compute the associated aggregate state. Section 2.5 already established that a stationary equilibrium exists under certain assumptions. One approach for constructing the solution could involve finding the stationary strategies of the agents and then computing the resulting aggregate state. However, a simpler alternative is to use

¹⁴Note that to highlight the difference between the algorithms they were implemented in a purely loop-based strategy. Vectorization could have led to substantial time reduction.

the optimal policy rules to calculate a distribution over the state space. Then, we derive the aggregates based on this distribution.

Definition 3.1 (Agent Distribution). Let P_z be the transition function of the exogenous shock z and g a given policy function. Let $P : X \times Z \times \mathcal{B}(X \times Z) \rightarrow [0, 1]$ be uniquely defined via

$$P_g((x, z), A \times B) \equiv P_z(z, B)\chi_A(g(x, z)). \quad (3.4)$$

Then P_g is a transition function for possible states $s \in S = X \times Z$.

For a probability measure $\mu \in \mathcal{P}(X \times Z)$ define the adjoint Markov operator

$$T_g\mu(A \times B) = \int P_g((x, z), A \times B)\mu(dx, dz) \quad (3.5)$$

We call the fixed point μ_g of T_g the stationary agent distribution.

The transition function $P_g(s, s')$ yields the probability of transitioning from a given state s today into the state s' tomorrow given that individuals behave according to the optimal decision rule $g(s)$. Hence, if current states are drawn from a given distribution μ , the adjoint operator $T_g\mu$ yields the distribution over states in the next period. Consequently, if μ is a fixed point of T_g , the distribution over states remains constant over time and thus, the aggregation of individual decision will also remain constant. This is an important step for finding a stationary equilibrium.

In most economic settings, with Assumptions 2.1 and 2.3 we already have that $\mu_x \times \mu_z$ is a fixed point of T_g , with μ_x denoting the invariant distribution over the endogenous state and μ_z the invariant distribution over exogenous shocks. However, in general we need to impose additional restrictions to ensure the existence of a unique stationary distribution (see, e.g., Kirkby, 2014).

Assumption 3.1 (Monotone Mixing Condition (MMC)). There exist $a, b \in S$ such that a is the lower bound of S and b is the upper bound of S . Also, there exists $\bar{s} \in S$ and $m \in \mathbb{N}$ such that $P_g^m(b, [a, \bar{s}]) > 0$ and $P_g^m(a, [\bar{s}, b]) > 0$.

Remark. Note that an interval $[x, y]$ on $S \subseteq \mathbb{R}^n$ refers to the subset given by $[x, y] = [(x_1, \dots, x_n), (y_1, \dots, y_n)] = \{s \in S : s_i \in [x_i, y_i] \text{ for } i = 1, \dots, n\}$

In words, this assumption states that regardless of starting from the highest or lowest state, there is always a positive probability of crossing the threshold \bar{s} within m or more periods. Also, note that Assumption 2.1 already ensures the existence of lower and upper bounds, since X and Z are assumed to be compact.

If the MMC condition is satisfied for our Markov process P_g , then we can show the following result from Hopenhayn and Prescott (1992), which provides us with a straightforward procedure of calculating the stationary distribution:

Theorem 3.4. *Suppose Assumption 3.1 is satisfied. If P_g is increasing with regards to the first-order stochastic dominance order \succeq , then there exists a unique stationary distribution μ_g for the Markov Process P_g . Additionally, for any initial distribution μ_0 , $T_g^n \mu_0 = \int P_g^n(s, \cdot) \mu_0(ds)$ converges to μ_g .*

Proof. We will follow the argument given in Hopenhayn and Prescott (1992):

Choose $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $P_g^m(b, [a, \bar{s}]) > \varepsilon$ and $P_g^m(a, [\bar{s}, b]) > \varepsilon$. Denote with δ_s the probability measure which puts mass 1 on the point set $\{s\}$.

Let $f : S \rightarrow \mathbb{R}_+$ be an arbitrary nonnegative, increasing, and continuous function. Then we have

$$\begin{aligned} \int f(s) T_g^m \delta_a(ds) &\geq f(a) \int_{s < \bar{s}} T_g^m \delta_a(ds) + f(\bar{s}) \int_{s \geq \bar{s}} T_g^m \delta_a(ds) \\ &\geq f(a)(1 - \varepsilon) + f(\bar{s})\varepsilon \\ &= \int f(s) \{(1 - \varepsilon)\delta_a + \varepsilon\delta_{\bar{s}}\}(ds) \end{aligned}$$

Hence, $T_g^m \delta_a \succeq (1 - \varepsilon)\delta_a + \varepsilon\delta_{\bar{s}}$ and with the same argument we see that $T_g^m \delta_b \preceq (1 - \varepsilon)\delta_b + \varepsilon\delta_{\bar{s}}$. Moreover, since T_g is increasing, so is T_g^k by induction and $T_g^m \delta_a \preceq T_g^m \delta_b$. Together we have established that

$$(1 - \varepsilon)\delta_a + \varepsilon\delta_{\bar{s}} \preceq T_g^m \delta_a \preceq T_g^m \delta_b \preceq (1 - \varepsilon)\delta_b + \varepsilon\delta_{\bar{s}}$$

and with monotonicity and linearity of T_g follows

$$(1 - \varepsilon)T_g^k \delta_a + \varepsilon T_g^k \delta_{\bar{s}} \preceq T_g^{m+k} \delta_a \preceq T_g^{m+k} \delta_b \preceq (1 - \varepsilon)T_g^k \delta_b + \varepsilon T_g^k \delta_{\bar{s}}$$

By Proposition 1 in Hopenhayn and Prescott (1992), the monotone sequences $\{T_g^k \delta_a\}$ and $\{T_g^k \delta_b\}$ will converge and we denote their limits with μ_a and μ_b . Also, there is a subsequence of $T_g^k \delta_{\bar{s}}$ converging to $\bar{\mu}$ and since the stochastic order induces closed graphs, we get:

$$(1 - \varepsilon)\mu_a + \varepsilon\bar{\mu} \preceq \mu_a \preceq \mu_b \preceq (1 - \varepsilon)\mu_b + \varepsilon\bar{\mu}.$$

But this inequality also implies that $\mu_a \preceq \bar{\mu} \preceq \mu_b$ which can only hold if $\mu_a = \bar{\mu} = \mu_b$. Now, $\bar{\mu}$ is a fixed point of T_g and from the monotonicity of T_g and the definition of μ_a and μ_b follows that $T_g^k \mu \rightarrow \bar{\mu}$ for any measure $\mu \in \mathcal{P}(S)$. □

With this result we can provide a simple algorithm for a numerical approximation of the stationary agent distribution. Following the notation from Kirkby (2019), for example, we can calculate μ_g as follows:

Algorithm 3.3 (Iteration on Measure of Agents).

Solve the individuals' optimization problem using Algorithm 3.1 or Algorithm 3.2.

Obtain the policy function $g(x, z)$.

Declare initial distribution μ_0 (array of size $n_x \times n_z$ whose elements sum to 1).

Declare iteration count $n = 0$.

Declare tolerance level ε .

while $\|\mu_{n+1} - \mu_n\| > \varepsilon$ **do**

 Set $n = n + 1$.

 Set $\mu_n = 0$.

for $x = 1, \dots, n_x$ **do**

for $z = 1, \dots, n_z$ **do**

for $z' = 1, \dots, n_z$ **do**¹⁵

$$\mu_n(x(g(x, z), z) = \mu_n(x(g(x, z), z')) + (g(x, z) * P(z, z')) * \mu_{n-1}(x, z)$$

end for

end for

end for

end while

As noted above, the agent distribution μ_g provides us with a measure over states when agents make decisions according to the rule g . If we denote with μ_i the probability distribution over X for one agent i resulting from the optimal strategy \mathbf{a}_i^* , then for any set $A \in \mathcal{B}(X)$ we will have

$$\mu_g(A) = \int_{i \in \mathcal{I}} \mu_i(A) d\lambda(i),$$

meaning that the measure of agents in a given set A will be equal to the aggregation of the individual probabilities of being in state A .

Example 3.2. For sake of simplicity assume we have a situation where $\mu_i(A) = i$, meaning that the probability for a given agent i to be in state A is equal to i . Then, the aggregate measure would yield $\mu_g(A) = 0.5$ and we would expect that exactly half of the population ends up in state A .

¹⁵Note that in this step we use the discretized version of the Markov transition function P_z and the operation $'*'$ denotes the convolution of two functions.

With this observation it is clear that there exists a mapping $\Phi : \mathcal{P}(S) \rightarrow \mathcal{Q}$ such that $\Phi(\mu_g) = \mathcal{A}((\mathbf{a}_i^*)_{i \in \mathcal{I}})$ and we can calculate the market aggregates by using μ_g instead of the optimal strategies.¹⁶ Therefore, we can now provide a simple algorithm for the computation of a stationary equilibrium:

Algorithm 3.4 (Computation of Stationary Equilibrium).

Declare tolerance level ε .

Declare initial aggregate state Q_0 .

Declare iteration count $n = 0$.

while $\|Q_{n+1} - Q_n\| > \varepsilon$ **do**

 Set $n = n + 1$.

 Set $Q_{old} = Q_{n-1}$

 Solve the individuals' problem given Q_{old} using Algorithm 3.1 or Algorithm 3.2.

 Compute the stationary agent distribution μ_p .

 Compute the implied aggregate state $Q_{new} = \Phi(\mu_p)$.

 Compute $d = \|Q_{new} - Q_{old}\|$

 Use appropriate root finding procedure to compute Q_n based on d .

end while

The last step in Algorithm 3.4 mentions using a root finding procedure, since the algorithm can be understood as finding the root of a function $F : \mathcal{Q} \rightarrow \mathbb{R}^q$ which yields the difference between a given market aggregate and the aggregate state from the optimal solution. For example, in a simple model economy as the one in Aiyagari (1994) discussed in Section 2.1, we could use a bijection method to find the equilibrium interest rate. The intuition for this is straightforward: If the model solution yields higher capital demand than capital supply, increase r and solve the model again. If capital supply exceeds capital demand, decrease r and solve the model again. This idea is illustrated in Figure 1a. In more complex settings with higher degrees of non-linearities, however, we might be forced to employ a more intricate strategy, such as a Quasi-Newton method.

¹⁶For example, in the model in Section 2.1 we were interested in total assets held by the consumers. Based on their optimal strategies this could be calculated by

$$A_1 = \int_{i \in \mathcal{I}} a_{i,t-1}(z_i^{t-1}) d\lambda(i).$$

However, if we know the distribution over states at time t and the optimal policy function, we could also calculate it as

$$A_2 = \int_{s \in \mathcal{S}} g(s) d\mu_g(s),$$

and $A_1 = A_2$.

4 Numerical Properties

With the results discussed in Sections 2 and 3, we now know under which conditions there exists an equilibrium in a BHA model economy and how to compute an equilibrium using full discretization of the state space. However, the numerical solution approach is merely an approximation of the true solution and convergence might not be guaranteed. Moreover, even if convergence is guaranteed, the numerical errors might still be huge for finite grid choices. Hence, in this section we will look more closely at the properties of the numerical solution algorithms discussed above. In doing so, we will primarily refer to the results in Kirkby (2017, 2019) which are based on the third chapter in Kirkby (2014).

4.1 Discretization Procedure

Before we can analyze the numerical solution properties, we follow the approach in Kirkby (2017) and formalize the discretization process that was briefly introduced in Section 3. We want to discretize the current endogenous state X , the decision variable Y and the exogenous shock Z . In the following we will use Assumption 2.1 as this ensures compactness of our sets.

First, we choose a number n_x and partition X into n_x mutually disjoint sets X_1, \dots, X_{n_x} such that $X = \cup_{i=1}^{n_x} X_i$. Then we select arbitrary points $x_i \in X_i$ for $i = 1, \dots, n_x$, and get a grid $\hat{X} = \{x_1, \dots, x_{n_x}\}$. We define the grid size of \hat{X} as

$$d_X = \max_{x \in X} \min_{\hat{x} \in \hat{X}} \|x - \hat{x}\|.$$

We use the same procedure for the choice variable Y and end up with a grid $\hat{Y} = \{y_1, \dots, y_{n_y}\}$ with associated grid size

$$d_Y = \max_{y \in Y} \min_{\hat{y} \in \hat{Y}} \|y - \hat{y}\|,$$

and for the exogenous shock we get $\hat{Z} = \{z_1, \dots, z_{n_z}\}$ with

$$d_Z = \max_{z \in Z} \min_{\hat{z} \in \hat{Z}} \|z - \hat{z}\|.$$

Finally, we discretize the transition function for Z using the same grid \hat{Z} and make sure that $\hat{P}(z, z_i) = P(z, Z_i)$, for all $i = 1, \dots, n_z$, with Z_i denoting the partition of Z that we used to form our grid. For example, this can be achieved using the method described in Tauchen (1986).

For our solutions we consider the set of piecewise constant functions

$$\hat{\mathcal{V}} = \left\{ \hat{v} : X \times Z \rightarrow \mathbb{R} \mid \begin{array}{l} \hat{v} \text{ is bounded, continuous, and} \\ \hat{v} \text{ is constant on } (X_i, Z_j), i = 1, \dots, n_x, j = 1, \dots, n_z \end{array} \right\}$$

As noted by Kirkby (2017), $\hat{\mathcal{V}}$ is a closed subspace of the set of bounded and continuous functions equipped with the norm $\|\hat{v}\| = \sup_{(x,z) \in X \times Z} |\hat{v}(x, z)|$ for $\hat{v} \in \hat{\mathcal{V}}$.

4.2 Error Bounds for the Approximated Value Function

To analyze the limiting behavior of our discrete algorithms provided in Section section 3, we now turn to computing upper bounds for the numerical errors committed by using a discretization procedure as described above. First, define the discretized version of the value function operator as used in Algorithm 3.3:

$$\hat{H}(v)(x_i, z_j) = \sup_{y \in \hat{\Gamma}(x_i, z_i)} R(x_i, y, z_i) + \beta \sum_{k=1}^{n_z} v(y, z_k) \hat{P}(z_j, z_k) \quad (4.1)$$

Similarly, the discretized version of the functional relationship determining the optimal value function is

$$\hat{v}(x_i, z_j) = \sup_{y \in \hat{\Gamma}(x_i, z_i)} R(x_i, y, z_i) + \beta \sum_{k=1}^{n_z} \hat{v}(y, z_k) \hat{P}(z_j, z_k) \quad (4.2)$$

Lemma 4.1. *With Assumption 2.1 the discretized Bellmans equation (4.2) has a unique solution $\hat{v} \in \hat{\mathcal{V}}$.*

Proof. With similar arguments as in the proof for Theorem 3.1, we note that \hat{H} satisfies Blackwell's sufficient conditions for a contraction (with modulus $\beta \in (0, 1)$), and the statement follows from the Contraction Mapping theorem. \square

As discussed in Section 2, the return function is bounded by Assumption 2.1 and the value function v will be bounded as well. Also, the following lemma will prove useful:

Lemma 4.2. *Let H be a contraction with fixed point v and let v_n denote the solution of iteratively applying H to an initial value v_0 . Then, for $\|v_n - v_{n-1}\| \leq \varepsilon$ we have*

$$\|v - v_n\| \leq \beta(1 - \beta)^{-1}\varepsilon.$$

The proof can be found in Appendix A. The value $\varepsilon > 0$ in the above lemma can be thought of as the tolerance level in one of our algorithms. In this case, v_n is the value

function that we end up with, if the algorithm terminates after n steps. Hence, we already have an upper bound for the distance between the discretized value function we get from our algorithm and the true discretized value function satisfying (4.2). Now, we only need a bound for the difference between the discretized value function and the true solution. Then, we could immediately infer an upper bound on the error we commit by using one of our algorithms.

The following result builds upon the concepts presented in Kirkby (2017). However, as our focus is solely on overall convergence, the derived error bounds are intentionally less restrictive to simplify the proof. Also, a similar analysis is provided by Santos and Vigo-Aguiar (1998).

Lemma 4.3. *Suppose Assumptions 2.1, 2.2, 2.3 and 3.1 are satisfied. Let v denote the optimal value function defined in (2.10) and \hat{v} denote the discretized value function defined in (4.2). Then for any $(x, z) \in X_i \times Z_j$ we have*

$$|v(x, z) - \hat{v}(x, z)| \leq c(i, j),$$

for a constant $c(i, j)$.

Proof. In the following, let x_{i-1} (x_{i+1}) denote the point in \hat{X} which is one grid point less (more) than x_i in every dimension (as X can have more than one dimension), and similarly for \hat{Y} and \hat{Z} .

Define the following constants:

$$\begin{aligned} d_{Ev}(i, j) &= \max_{x_l \in \hat{X}} \sum_{k=1}^{n_z} |v(x_l, z_{j+1}) - v(x_l, z_{j-1})| \hat{P}(z_j, z_k) \\ d_v(i, j) &= \max \left\{ |v(x_{i+1}, z_{j+1}) - v(x_i, z_j)|, |v(x_{i-1}, z_{j-1}) - v(x_i, z_j)| \right\} \\ d_R(i, j) &= \max_{y \in \hat{Y}} \max \left\{ |R(x_i, y_k, z_j) - R(x_i, y_{k-1}, z_j)|, \right. \\ &\quad \left. |R(x_i, y_k, z_j) - R(x_i, y_{k+1}, z_j)| \right\}, \end{aligned}$$

which are well-defined and finite, since R and v are bounded. Then we have

$$|v(x, z) - v(x_i, z_j)| \leq d_v(i, j),$$

and

$$\begin{aligned}
|v(x_i, z_j) - \hat{v}(x_i, z_j)| &= \left| \sup_{y \in \Gamma(x_i, z_j)} R(x_i, y, z_j) + \beta \int v(y, z') P(z_j, dz') \right. \\
&\quad \left. - \sup_{y \in \hat{\Gamma}(x_i, z_j)} R(x_i, y, z_j) + \beta \sum_{k=1}^{n_z} \hat{v}(y, z_k) \hat{P}(z_j, z_k) \right| \\
&\leq \left| \sup_{y \in \Gamma(x_i, z_j)} R(x_i, y, z_j) - \sup_{y \in \hat{\Gamma}(x_i, z_j)} R(x_i, y, z_j) \right| \\
&\quad + \left| \sup_{y \in \Gamma(x_i, z_j)} \beta \int v(y, z') P(z_j, dz') - \sup_{y \in \hat{\Gamma}(x_i, z_j)} \beta \sum_{k=1}^{n_z} \hat{v}(y, z_k) \hat{P}(z_j, z_k) \right| \\
&\leq d_R(i, j) + \beta d_{Ev}(i, j)
\end{aligned}$$

Hence, with the triangular inequality it follows that

$$\begin{aligned}
|v(x, z) - \hat{v}(x, z)| &\leq |v(x, z) - v(x_i, z_j)| + |v(x_i, z_j) - \hat{v}(x_i, z_j)| \\
&\leq d_v(i, j) + d_R(i, j) + \beta d_{Ev}(i, j) \equiv c(i, j).
\end{aligned}$$

□

This result tells us that the difference between the correct solution and our approximated solution will only depend on the partition of the state space and converge to zero as the distance between the grid points goes to zero (as this would mean that the constants defined in the proof would go to zero). Now, we can finally calculate an upper bound for the numerical error $\|v - \hat{v}_N\|$ resulting from using our discrete algorithm when using a given tolerance level as stopping criterion.

Theorem 4.4. *Suppose Assumptions 2.1, 2.2, 2.3 and 3.1 are satisfied. Let v denote the optimal value function defined in (2.10) and \hat{v}_n the sequence of discretized value functions obtained by iteratively applying \hat{H} starting from an initial function v_0 . If \hat{v}_N denotes the discretized value function at which the algorithm stops given the stopping criterion $\|v_n - v_{n-1}\| < \varepsilon$, for some $\varepsilon > 0$, then*

$$\|v - \hat{v}_N\| \leq \frac{\beta}{1 - \beta} (C + \varepsilon),$$

where

$$C = \frac{1 - \beta}{\beta} \max_{\substack{i=1, \dots, n_x, \\ j=1, \dots, n_z}} c(i, j)$$

Proof. Let \hat{v} be the fixed point of \hat{H} . The statement immediately follows from the

triangular inequality together with lemmas 4.2 and 4.3:

$$\begin{aligned} \|v - \hat{v}_N\| &\leq \|v - \hat{v}\| + \|\hat{v} - \hat{v}_N\| \\ &\leq \|c(i, j)\| + \beta(1 - \beta)^{-1}\varepsilon. \end{aligned}$$

□

4.3 Error Bounds for the Approximated Policy Rule

The results above provide us with an upper bound on the numerical error in the value function. Next, we are also interested in the optimal policy resulting from the approximated value function. Let \hat{v}_N denote the approximation of the true value function resulting from our discrete algorithms after N iterations. Denote with \hat{g} the associated policy rule resulting from the algorithm and with g the correct optimal policy. What can we say about $\|g - \hat{g}\|$?

To answer this question we first need to put even more structure on our return function.

Assumption 4.1. For any given choice of x and z , the return function $R(x, y, z)$ is decreasing in y .

This assumption is again not very restrictive. Remember that the current choice variable y becomes next period's state variable x . In most economic settings, a higher state variable next period corresponds to a higher continuation value, while reducing current period's pay-offs. For example, consider once again the simple economy in Section 2.1. There, the choice variable is asset holdings and while higher savings would lead to higher welfare next period, they reduce current returns, since higher savings induce lower consumption. In contrast, suppose the assumption does not hold for a specific model. Then, agents could choose a higher variable y to increase both their current and future returns. Such a problem would not be well-posed and agents would always choose the highest possible y . Hence, in a BHA economy this assumption is almost always satisfied.

To find the error bounds on our approximated policy function, we again follow a simplified version of the ideas presented in Kirkby (2017). First, let $g_{\pm m}(x, z)$ denote m grid points more or less than $g(x, z)$ in every dimension (again, keep in mind that the

choice set may have more than one dimension), and define the following expressions:

$$d_{g,m}(i, j) = \max \left\{ |g_{+m}(x_i, z_j) - g(x_i, z_j)|, |g_{-m}(x_i, z_j) - g(x_i, z_j)| \right\}$$

$$d_{R,m}(i, j) = \max \left\{ |R(x_i, g_{+m}(x_i, z_j), z_j) - R(x_i, g(x_i, z_j), z_j)|, \right. \\ \left. |R(x_i, g_{-m}(x_i, z_j), z_j) - R(x_i, g(x_i, z_j), z_j)| \right\}$$

Now we can describe an upper bound for the error in the policy function that results from using a discrete algorithm and an approximated value function.

Theorem 4.5. *Suppose Assumptions 2.1, 2.2, 2.3, 3.1 and 4.1 hold. Let g denote the optimal policy function and \hat{g} the approximated policy function resulting from our discrete algorithm after N iteration. Assume that for the associated value functions we have $\|v - \hat{v}_n\| \leq \delta_V$. Then for any $(x, z) \in X_i \times Z_j$ we have*

$$|g(x, z) - \hat{g}(x, z)| \leq d_g(x, z) \quad \text{almost everywhere,}$$

with

$$d_g(x, z) = \max \left\{ \sum_{m=0}^{l_+} d_{g,m}, \sum_{m=0}^{l_-} d_{g,m} \right\}$$

$$l_+ = \arg \min \left\{ \sum_{m=0}^l d_{R,m}(x, z) \leq \delta_V \right\}$$

$$l_- = \arg \min \left\{ \sum_{m=0}^l d_{R,m}(x, z) \geq \delta_V \right\}$$

Proof. Remember that the correct optimal policy function is defined as

$$g(x, z) = \arg \sup_{y \in \Gamma(x, z)} R(x, y, z) + \beta \int v(y, z') P(z, dz'), \quad (4.3)$$

while our approximated policy function takes the form

$$\hat{g}(x, z) = \arg \sup_{y \in \hat{\Gamma}(x, z_j)} R(x_i, y, z_j) + \beta \sum_{k=1}^{n_z} \hat{v}_N(y, z_k) \hat{P}(z_j, z_k). \quad (4.4)$$

Note that the suprema on the right side are just the value functions $v(x, z)$ and $\hat{v}_{N+1}(x, z)$. Hence, the suprema can differ by at most δ_V at any point.

First, assume $g(x, z) > \hat{g}(x, z)$. We want to find out by how much g can exceed \hat{g} . For any choice $\tilde{g}(x, z) > \hat{g}(x, z)$, the return function will decrease. Assuming that $\tilde{g}(x, z)$ lies

within l grid points of $\hat{g}(x, z)$, the associated loss in the return function is

$$R(x, \tilde{g}(x, z), z) - R(x, g(x, z), z) \leq - \sum_{m=0}^l d_{R,m}(x, z).$$

However, $\tilde{g}(x, z)$ can only fulfill (4.3), if the attained supremum is at least as great as that attained by \hat{g} . Thus, it must hold that

$$\delta_V \geq - \sum_{m=0}^l d_{R,m}(x, z).$$

Denote with l_+ the minimum l for which this condition is satisfied, that is

$$l_+ = \arg \min_{l \geq 0} \left\{ \delta_V \geq - \sum_{m=0}^l d_{R,m}(x, z) \right\}.$$

Then l_+ represents the maximum distance we have to go and the corresponding distance in g is given by

$$d^+ = \sum_{m=0}^{l_+} d_{g,m}$$

With same argument for $\tilde{g}(x, z) < \hat{g}(x, z)$ we get a bound l_- and a distance d^- . Hence, the statement holds with

$$d_g(x, z) = \max\{d^+, d^-\}.$$

Lastly, the result holds only almost everywhere, since g is the optimal policy only almost everywhere. □

4.4 Error Bound for Approximated Agent Distribution

Our algorithm to compute the stationary equilibrium, Algorithm 3.4, not only uses the solutions to the individuals' optimization problems, but also the aggregation of individual strategies through the invariant distribution over states. Before turning to the main result of convergence towards the true equilibrium, we still need to look at the difference between our approximated agent distribution and the true distribution.

For the following analysis and to reproduce the arguments presented in Kirkby (2019), it will be convenient to represent the Markov operator for the agent distribution in an iterated function systems notation. Remember that our transition function between states

was defined as

$$P_g((x, z), A \times B) \equiv P_z(z, B)\chi_A(g(x, z)), \quad (4.5)$$

where P_z is the transition function on for the Markov process on Z . Now, consider a function $\phi : S \times W \rightarrow S$ and the i.i.d. random variable $\Omega : (W, \mathcal{B}(W)) \rightarrow \mathbb{R}$. Then, our transition function can also be represented as

$$P(s, C) = \Omega(\{\omega : \phi(s, \omega) \in C\}), \quad (4.6)$$

for any $s \in S$ and $C \in \mathcal{B}(S)$ (see, e.g., Stenflo, 2001; Santos and Peralta-Alva, 2005).¹⁷

Also, we introduce a new metric to indicate the distance between two measures. Let \mathcal{F} denote the set of bounded continuous functions $f : S \rightarrow [0, 1]$. Then, for two measures $\mu, \nu \in \mathcal{P}(S)$ we define

$$\|\mu - \nu\| = \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : f \in \mathcal{F} \right\}$$

Now we have the tools to find an upper bound for the numerical error that we introduce by using the discretized policy function and the discretized Markov operator for the calculation of the agent distribution. First, the following lemma from Kirkby (2019) provides us with a distance measure for the resulting transition functions:

Lemma 4.6. *Let $g, \hat{g} : S \rightarrow X$ be two policy functions satisfying $\|g - \hat{g}\| \leq \delta$. Denote with P the transition function associated with g as defined in (4.5) and with \hat{P} the transition function associated with \hat{g} and the discretized exogenous shock \hat{P}_z , as used in Algorithm 3.3. Then*

$$\|P^m - \hat{P}^m\| \leq m\delta$$

Proof. We first look at \tilde{P} , which is the transition function that is derived from using \hat{g} together with the true P_z . We have:

¹⁷For example, if $P_z(z, B) = \pi(z)$, we could find an i.i.d shock ω such that $\Omega((z + \omega) \in B) = \pi(z)$.

$$\begin{aligned}
& \|P^m - \tilde{P}^m\| \\
&= \max_{s \in S} \left[\int \|P^m(s, \omega) - \tilde{P}^m(s, \omega)\| \Omega(d\omega) \right] \\
&\leq m \max_{s \in S} \left[\int \|P(s, \omega) - \tilde{P}(s, \omega)\| \Omega(d\omega) \right] \\
&= m \max_{s \in S} \left[\int \|(g(s), P_z(z, dz(d\omega))) - (\hat{g}(s), P_z(z, dz(d\omega)))\| \Omega(d\omega) \right] \\
&\leq m \max_{s \in S} \left[\int \|g(s) - \hat{g}(s)\| \Omega(d\omega) \right] \\
&\leq m \max_{s \in S} \left[\int \delta \Omega(d\omega) \right] \\
&= m\delta
\end{aligned}$$

Next, remember that our discretized transition matrix operates on the partition $X_i \times Z_j$ and that $\hat{P}_z(z, z_i) = P(z, Z_i)$ for all $i = 1, \dots, n_z$. Thus, we have

$$\begin{aligned}
& \|\tilde{P}^m - \hat{P}^m\| \\
&= \max_{s \in S} \left[\int \|\tilde{P}^m(s, \omega) - \hat{P}^m(s, \omega)\| \Omega(d\omega) \right] \\
&\leq m \max_{s \in S} \left[\int \|\tilde{P}(s, \omega) - \hat{P}(s, \omega)\| \Omega(d\omega) \right] \\
&= m \max_{s \in S} \left[\int \|(\hat{g}(s), P_z(z, dz(d\omega))) - (\hat{g}(s), \hat{P}_z(z, dz(d\omega)))\| \Omega(d\omega) \right] \\
&= m \max_{s \in S} \left[\sum_{\substack{i=1, \dots, n_x \\ j=1, \dots, n_z}} \int_{X_i \times Z_j} \|(\hat{g}(s), P_z(z, dz(d\omega))) - (\hat{g}(s), \hat{P}_z(z, dz(d\omega)))\| \Omega(d\omega) \right] \\
&= m \max_{s \in S} \left[\sum_{\substack{i=1, \dots, n_x \\ j=1, \dots, n_z}} \int_{X_i \times Z_j} 0 \Omega(d\omega) \right] \\
&= 0
\end{aligned}$$

Finally, with the triangular inequality we get

$$\|P^m - \hat{P}^m\| \leq \|P^m - \tilde{P}^m\| + \|\tilde{P}^m - \hat{P}^m\| \leq m\delta$$

□

This lemma tells us that that the numerical error in our transition matrix resulting from our discretization procedure is directly related to the error bound from the discretized policy functions. In particular, if our approximated policy function converges to the true policy function, then the error in our approximated transition function will vanish as well.

Now, the last result we need is to translate this error bound from the transition function to the stationary agent distribution. Following Kirkby (2019) we get the following lemma:

Lemma 4.7. *Suppose Assumption 2.1 holds and the adjoint Markov operator of our transition function, $T^{m,*}$, is a contraction mapping with modulus β . If $\|g - \hat{g}\| < \delta$, for some $\delta > 0$, then*

$$\|\mu^* - \hat{\mu}^*\| \leq \frac{m\delta}{1 - \beta}$$

Proof. From Lemma 4.6 we get $\|P^m - \hat{P}^m\| < m\delta$. Now, for any Lipschitz continuous function f with constant L we get

$$\begin{aligned} & \left| \int f(s)\mu^*(ds) - \int f(s)\hat{\mu}^*(s) \right| \\ &= \left| \int f(s)P^m \cdot \mu^*(ds) - \int f(s)\hat{P}^m \cdot \hat{\mu}^*(s) \right| \\ &\leq \left| \int f(s)P^m \cdot \mu^*(ds) - \int f(s)P^m \cdot \hat{\mu}^*(s) \right| \\ &\quad + \left| \int f(s)P^m \cdot \hat{\mu}^*(s) - \int f(s)\hat{P}^m \cdot \hat{\mu}^*(s) \right| \\ &\leq \left| \int f(s)P^m \cdot \mu^*(ds) - \int f(s)P^m \cdot \hat{\mu}^*(s) \right| + L\|P^m - \hat{P}^m\| \\ &\leq \beta \left| \int f(s)\mu^*(ds) - \int f(s)\hat{\mu}^*(s) \right| + Lm\|P^m - \hat{P}^m\| \\ &\leq \beta \left| \int f(s)\mu^*(ds) - \int f(s)\hat{\mu}^*(s) \right| + Lm\delta \end{aligned}$$

Since S is compact by assumption, we can define $d = \sup_{s,s'} \|s - s'\|$, which is finite. If $\|f\| \leq 1$, then f is Lipschitz with $L \leq 1/d$.

Hence, the result follows from

$$\|\mu^* - \hat{\mu}^*\| = \sup \left\{ \left| \int f(s)\mu^*(ds) - \int f(s)\hat{\mu}^*(s) \right| : f \in \mathcal{F} \right\},$$

□

Thus, we have established that our discretization procedure adds a numerical error to our stationary agent distribution which is bounded above by the error from the discretized policy function. Lastly, we have to consider that our algorithm usually stops before attaining the fixed point of the operator. Hence, summarizing all the results above, we find:

Theorem 4.8. *Suppose Assumptions 2.1 and 3.1 hold. Let μ^* be the true stationary agent distribution and $\hat{\mu}_N$ the approximated distribution resulting from Algorithm 3.3 with the stopping criterion $\|\hat{\mu}_{N+m} - \hat{\mu}_N\| \leq \varepsilon$. Also, the approximation of the policy function satisfies $\|g - \hat{g}\| < \delta$. Then*

$$\|\mu^* - \hat{\mu}_N\| \leq (1 - \beta)^{-1}(m\delta + \beta\varepsilon)$$

Proof. With the triangular inequality we get

$$\|\mu^* - \hat{\mu}_N\| \leq \|\mu^* - \hat{\mu}\| + \|\hat{\mu} - \hat{\mu}_N\|$$

where $\hat{\mu}$ is the fixed point of the discretized operator. Then the result immediately follows from Lemma 4.7 and Lemma 4.2. □

4.5 Summary of Results

Now, we finally have all the ingredients to characterize the convergence of our discretized algorithms. In particular, we know under which conditions the approximated solutions will converge to the true solution as the grid size and the tolerance level of the algorithms become smaller:

Proposition 4.9. *Let Assumptions 2.1, 2.2, 2.3, 3.1 and 4.1 be satisfied. Let \hat{v}_N denote the numerical solution to the discretized value function problem attained after N iterations. Then, the numerical errors in the value function, $\|v - \hat{v}_N\|$, in the associated optimal policy function, $\|g - \hat{g}_N\|$, and in the stationary agent distribution, $\|\mu^* - \hat{\mu}_N\|$, converge to zero as the distance between the grid points in the dimensions being discretized go to zero almost everywhere and as $N \rightarrow \infty$.*

Proof. Since R and V are bounded functions on compact sets, the difference of evaluating them at any two adjacent grid points must go to zero as the distance between the grid points goes to zero. The result therefore follows from theorems 4.4, 4.5 and 4.8. □

Corollary 4.10 (Convergence to Equilibria). *Let $F : \mathcal{Q} \rightarrow \mathbb{R}$ be the function whose image is the difference between a given aggregate state and the aggregation of individual actions. Let \hat{F} denote the associated discretized function as computed in Algorithm 3.4. Under the assumptions of Proposition 4.9, $\|F - \hat{F}\|$ converges to zero. In particular, if F is well behaved and possesses a root in \mathcal{Q} , Algorithm 3.4 will converge to an equilibrium of the model.*

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Appendices

A Additional Proofs

Proof of Lemma 2.1

Proof. First, we follow the idea presented in Stokey et al. (1989b) to show that the set of feasible strategies is nonempty. For any $i \in \mathcal{I}$ fix any initial condition $s_{i,0} = (x_{i,0}, z_{i,0})$. Assumption 2.1 ensures that Γ is compact-valued and continuous. Also, X and Z are subsets of Euclidean spaces. Hence, Γ permits a measurable selection (see, e.g., Hildenbrand, 1974), meaning that there exists a measurable function $h : S \rightarrow X$ such that $\forall s_t \in S : h(s_t) \in \Gamma(s_t; Q_t)$. Now define a strategy \mathbf{a}_i as

$$\begin{aligned} a_{i,0} &= h(s_{i,0}) \\ a_{i,t}(z^t) &= h(a_{i,t-1}(z^{t-1}), z_t), \quad z^t \in Z^t, t = 1, 2, \dots \end{aligned}$$

Now, $a_{i,0}$ is clearly measurable and since compositions of measurable functions are again measurable, $a_{i,t}$ is also measurable for any t by induction. Hence, \mathbf{a}_i is a feasible strategy as in Definition 2.4. In particular, for all $i \in \mathcal{I}$ the set of feasible strategies is non-empty.

Next, consider the optimization problem

$$\sup_{\mathbf{a}_i \in \mathfrak{N}_i(s_{i,0}; \mathbf{Q})} R(x_{i,0}, a_{i,0}(z_{i,0}), z_{i,0}) + \mathbb{E}_0 \left\{ \sum_{t=1}^{\infty} \beta^t R(a_{i,t-1}(z_i^{t-1}), a_{i,t}(z_i^t), z_{i,t}) \right\}.$$

Since R is bounded, the supremum exists and will be attained for some $\mathbf{a}^* = \lim_{k \rightarrow \infty} \mathbf{a}^k$ with $\{\mathbf{a}^k\}_{k=1}^{\infty} \in \mathfrak{N}(s_0; \mathbf{Q})$. And since the constraint Γ is compact-valued and continuous, $\mathbf{a}^* = \lim_{k \rightarrow \infty} \mathbf{a}^k \in \mathfrak{N}(s_0; \mathbf{Q})$. Hence, \mathbf{a}^* is feasible and thus an optimal strategy. □

Proof of Lemma 2.4

Proof. The proof follows Acemoglu and Jensen (2015) and uses the results by Topkis (1978, 1998) on optimization with lattices and supermodular functions.

Fist, note that $v^*(x, z)$ can be obtained by iterating on the value function as in Equation 3.1. Since supermodularity is preserved by integration (and hence by taking the expectation with respect to the transition function of z), $v^*(x, z)$ will be supermodular in x if $R(x, y, z)$ is supermodular in (x, y) and $\Gamma(\cdot, z)$ is a sub-lattice of $X \times X$.

Now, remember that the policy correspondence is defined as

$$\begin{aligned} G(x, z) &= \left\{ y \in \Gamma(x, z) : \right. \\ &\quad \left. v^*(x, z) = R(x, y, z) + \beta \mathbb{E} \left[v^*(y, z') \right] \right\} \\ &= \arg \sup_{y \in \Gamma(x, z)} R(x, y, z) + \beta \mathbb{E} \left[v^*(y, z') \right], \end{aligned}$$

and the result immediately follows from Topkis's theorem, since $R(x, y, z) + \beta \mathbb{E} \left[v^*(y, z') \right]$ is again supermodular. \square

Proof of Lemma 2.5

Proof. We look at $(\mathcal{P}(X), \succeq)$ with \succeq being the first-order stochastic dominance order and show that for two measures $\mu_2 \succeq \mu_1$ we have $T_{g_i, Q}^* \mu_2 \succeq T_{g_i, Q}^* \mu_1$. This holds if and only if for every increasing function $f : X \rightarrow \mathbb{R}$ we have:

$$\int f(x) T_{g_i, Q}^* \mu_2(dx) \geq \int f(x) T_{g_i, Q}^* \mu_1(dx) \quad (\text{A7})$$

Remember that

$$T_{g_i, Q}^* \mu = \int P_{g_i, Q}(x, \cdot) \mu(dx),$$

and

$$P_{g_i, Q}(x, A) \equiv \mu_z(\{z \in Z \mid g_i(x, z; Q) \in A\}),$$

with μ_z the invariant distribution over z and g_i the selection from the optimal policy correspondence. Hence, Equation A7 is equivalent to

$$\int \left[\int f(g_i(x, z; Q)) \mu_2(dx) \right] \mu_z(dz) \geq \int \left[\int f(g_i(x, z; Q)) \mu_1(dx) \right] \mu_z(dz). \quad (\text{A8})$$

Now, with Lemma 2.4 we have that $f \circ g_i$ is an increasing function in x and Equation A8 will always hold whenever $\mu_2 \succeq \mu_1$. \square

Proof of Lemma 4.2

Proof. With the triangular inequality we get

$$\begin{aligned}
\|v_n - v_{n+m}\| &\leq \sum_{i=1}^m \|v_{n+i-1} - v_{n+i}\| \\
&\leq \sum_{i=1}^m \beta^{n+i-1} \|v_0 - v_1\| \\
&= \beta^n \|v_0 - v_1\| \sum_{i=1}^{m-1} \beta^i \\
&\leq \beta^n \|v_0 - v_1\| \sum_{i=1}^{\infty} \beta^i = \beta^n \|v_0 - v_1\| \frac{1}{1-\beta}
\end{aligned}$$

With $m \rightarrow \infty$, we get $\|v_n - v\| \leq \beta^n (1-\beta)^{-1} \|v_0 - v_1\|$. And finally, we can look at a new sequence $\tilde{v}_0 = v_{n-1}$, $\tilde{v}_1 = v_n$ and get

$$\begin{aligned}
\|v_n - v\| &= \|\tilde{v}_1 - v\| \leq \beta(1-\beta)^{-1} \|\tilde{v}_0 - \tilde{v}_1\| \\
&= \beta(1-\beta)^{-1} \|v_{n-1} - v_n\| \\
&= \beta(1-\beta)^{-1} \varepsilon.
\end{aligned}$$

□

B Additional Theorems

Theorem B11 (Theorem 1 in Hopenhayn and Prescott, 1992). *Let Λ be a compact subset of $\mathcal{P}(S)$ and $T : \Lambda \rightarrow \Lambda$ an increasing map. Then, T has a fixed point if and only if there exists a measure $\mu_a \in \Lambda$ such that $T\mu_a \succeq \mu_a$.*

Proof. See Hopenhayn and Prescott (1992). □

Theorem B12 (Blackwell's sufficient conditions for a contraction). *Let $X \subseteq \mathbb{R}^l$, and let $B(X)$ be the space of bounded functions $f : X \rightarrow \mathbb{R}$, with the supremum norm. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying*

(i). *Monotonicity: For all $f, g \in B(X)$, if $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*

(ii). *Discounting: $\exists \beta \in (0, 1) : \forall f \in B(X), a \in \mathbb{R}_+, x \in X:$*

$$\left[T(f + a)(x) \right] \leq (Tf)(x) + \beta a,$$

Then T is a contraction mapping with modulus β .

Proof. For all $f, g \in B(X)$, with $f \leq g$, we have $f \leq g + \|f - g\|$. With properties (i) and (ii) we get

$$Tf \leq T\left(f \leq g + \|f - g\|\right) \leq Tg + \beta\|f - g\|$$

By applying the same logic for $g \leq f$, we get

$$Tg \leq Tf + \beta\|f - g\|,$$

and combining the two inequalities we get

$$\|Tf - Tg\| \leq \beta\|f - g\|$$

□

Theorem B13 (Theorem of the Maximum). *Let $X \subseteq \mathbb{R}^l$, $Y \subseteq \mathbb{R}^m$, and $Z \subseteq \mathbb{R}^k$, $f : X \times Y \times Z \rightarrow \mathbb{R}$ a continuous function, and $\Gamma : X \times Z \rightarrow Y$ a compact-valued and continuous correspondence. Then the value function $v : X \times Z \rightarrow \mathbb{R}$ defined in Equation 2.10 is continuous, and the correspondence $G : X \times Z \rightarrow \mathbb{R}$ defined in Equation 2.11 is non-empty, compact-valued and upper hemicontinuous.*

Proof. For a proof see Stokey et al. (1989a). □

Theorem B14 (Contraction Mapping Theorem). *If (S, ρ) is a complete metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β , then*

(i). *T has exactly one fixed point $v \in S$*

(ii). *For any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 1, 2, \dots$*

Proof. For a proof see, for example, Stokey et al. (1989a). □

Theorem B15 (Kakutani-Glicksberg-Fan Fixed-Point Theorem). *Let S be a non-empty, compact and convex subset of a Hausdorff locally convex topological vector space. If $\Phi : S \rightarrow 2^S$ is upper hemicontinuous and $\Phi(s)$ is non-empty, compact and convex for all $s \in S$, then Φ has a fixed point.*

Proof. See the works by Kakutani (1941), Glicksberg (1952) and Fan (1952). □

C Parameterization for Model in Section 2.1

To produce the graphs in Figure 1, the model is solved numerically in MATLAB 2021b by using value function iteration on a discretized state space. It is assumed that the logarithm

of the labor supply shock follows an AR(1) process:

$$\log(h_{t+1}) = \rho \log(h_t) + \varepsilon_t$$

with $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. This process is approximated by a seven-state Markov chain using the procedure in Tauchen (1986).

Preferences of households are given by a utility function with constant relative risk-aversion:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

The final good is produced using a Cobb-Douglas production function:

$$Y = K^\alpha L^{1-\alpha},$$

with the capital share α chosen to be 0.36 and capital depreciates at rate $\delta = 0.04$.

A summary of the parameter values is given in Table C1.

Table C1: Parameterization

	Parameter	Value
Risk aversion	σ	2
Discount factor	β	0.96
Capital Share in Production	α	0.36
Depreciation rate	δ	0.04
Autocorr. labor endowment	ρ	0.9
White noise std. dev.	σ_ε	0.4